

On the optimal control of an animal-vegetation ecological system

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Abstract

This paper studies the optimal control of an animal-vegetation system where growth in the animal stock is density independent. The ecological system is shown to have a limit cycle, under certain parameter values, but can also be stabilized by a simple harvesting rule. The managing unit maximizes profits from selling the meat, over an infinite planning horizon. With a domestic animal stock, costs are related to keeping a large animal stock, and not to harvesting. An optimal steady state is described, and the effects of parameter changes is discussed, in particular the discount rate. The effect of discounting on the optimal stock size is demonstrated to be ambiguous. The optimal approach path is derived analytically, and is shown to be a saddle path in state-state space. The model is exemplified with a numerical analysis that compares the discounted profit from the optimal harvest regime to a management regime implying a constant harvest ratio. Finally, a positive stock effect is discussed.

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1. Introduction

Sharp conclusions from bioeconomic models can be hard to obtain when more than one species is included. Nevertheless, multidimensional systems deserve attention as established insights from single-species models frequently do not carry over to situations where several species interact. In this paper the optimal management of a grazing model is examined, where animals and vegetation together constitute a two-species ecological system to be controlled by a single management unit. The ecological interaction is modelled as a predator-prey system, where in our context the animals are the predator and vegetation is the prey. Such systems have some well-known characteristics, and in particular, are able to exhibit deterministic equilibrium cycles; see e.g. Rosenzweig and MacArthur (1963) and May (1975) for general predator-prey models, or Caughley (1976) for grazing systems. The system studied here is cyclical under certain parameter values, and stability properties are dependent on the harvesting policy (as in Brauer and Soudack, 1979). Using a model originally formulated by Noy-Meir (1972), which is linear in the animal stock but concave in vegetation, analytical results are obtained with respect to the optimal steady state and also the approach path. The results are further demonstrated by a numerical example.

The management unit is assumed to maximize discounted profits generated by slaughtering the animals and selling the meat. Revenues are defined as the quantity of meat sold times a unit price, which is assumed to be constant over time and unaffected by the harvest decision. Livestock management also involves costs, which are mainly associated with herding. Consequently, because the animals here are domestic, and in contrast with the typical fishery model, costs are increasing with the size of the stock and independent of harvest. The planning horizon is infinite, meaning that we are looking for optimal steady states. We find that an optimal steady state can be identified, and the sensitivity of the optimal steady state to parameter changes is examined; in particular the discount rate. It is shown that the steady state animal stock may go up or down with discounting, and the general result is parameter dependent. Next, we derive the optimal approach, and find that it follows a saddle path in state-state space, after an initial ‘most rapid approach’ period where harvest is set to its maximum or minimum. Finally, the implications of a positive stock effect are discussed.

The present study is partly motivated by challenges faced by the Saami reindeer herding community in the Northern parts of Scandinavia. The reindeer herding community has

experienced a recent history of large fluctuations in vegetation quality and stock sizes, implying long periods of small and/or unstable animal stocks (Johannessen and Skonhoft 2000), which has led many author to focus on common property mechanisms and strategic overstocking in a static framework. In contrast, this paper puts emphasis on the dynamic properties of the underlying ecosystem, and the implications for optimal management. The model may also be representative for livestock management in other semi-arid regions, such as the Sahel area. Several studies focus on the self-subsistence element of pastoralism, and model herders as risk averse agents that maximize a concave utility function, possibly with a lower constraint on consumption. However, in Scandinavia meat is produced mainly for sale and herders have access to credit markets and insurance. Consequently the managing unit here is modelled as a pure profit maximizer, except from in the last section where a positive stock effect is discussed.

Almost all contributions on predator-prey systems within the economics literature are based on models where growth in both species is logistic. A stability analysis of optimal steady states in such predator prey-models is found in Silvert and Smith (1973). Pioneering work in the fisheries literature regarding optimal multi-species harvesting includes Anderson (1975) and Hannesson (1983), who consider the properties of optimal steady states depending on parameter values, with, in the latter paper, numerical simulations to describe optimal approach paths. This is also done by Perrings (1997), in the case of a grazing system. Wilen and Brown (1986) manage to obtain a closed form solution to the optimal control path, in a fishery model with one way interaction, so that the abundance of prey is unaffected by predation. Semmler and Sieveking (1984) examine the optimal control problem with interacting species more generally, and emphasize that certain established truths from one species models do not automatically carry over to the multi-species case, both with respect discounting and existence of optimal controls More recent contributions include Hoekstra and van den Berg (2005), who characterize alternative steady states, and Li and Løfgren (1998), who, in addition, derive the stability properties more rigorously.

The present paper combines insights form Ragozin and Brown (1985), who derive the control path analytically in a mutually logistic predator-prey model, and Brekke, et. al. (2007) who use a model similar to the one presented here, but resort to numerical methods. The main contribution is first to derive the optimal control path analytically in the case of a possibly unstable ecological system. Second, and in contrast to earlier contributions, to use an

objective function that is suitable for the management of a domestic animal stock, and therefore differs from the typical fisheries or wildlife management situation. Finally, we also discuss the implications of a positive stock effect.

The rest of the paper is organised as follows. Section 2 presents and analyses the ecological model, in absence of man. Animal harvesting is introduced in section 3, and the implications for stability and yield for a given harvest rate is analysed. While no economic motives are present here, the harvest problem in section 4 is based on a profit function that is maximized. The optimal harvest schedule in steady state is described, whereas Section 5 is devoted to the optimal approach path. Section 6 presents a numerical illustration. Section 7 analyzes the implications of including a positive stock effect. The last section 8 summarizes our findings.

2. The ecological model

The ecological predator-prey model employed here is equivalent to the herbivore-vegetation model formulated by Noy-Meir (1975), and is what Caughley and Lawton (1981) would classify as a *lassiez faire* model, where the herbivores do not interfere with each other in a way such that the animal stock growth rate is affected by the stock size; the only thing that affects the animal stock growth rate is food abundance. The underlying assumption is that the animals are not territorial, so that space per animal does not affect fertility. Alternatively; the animal stock is limited by food availability to such an extent that other factors affecting herd growth can be neglected. In contrast, in an *interferential* model the growth function for animals would be modelled as logistic, while vegetation abundance plays the role of ‘carrying capacity’ for the animals. Such models include the ‘generalized’ Lotka-Volterra model (as used, e.g., by Hannesson, 1983), or the similar Leslie-Gower model (Leslie, 1948). These two types of models have different dynamics, since models with logistic growth in both species are globally stable, as opposed to the model used on this paper which exhibits a limit cycle.

The model does not distinguish between age classes; both animals and vegetation are formulated simply as ‘biomass’. The model is set in continuous time, and is completely deterministic, so that stability issues are focused upon with a minimum of mathematical resistance.

The growth equations for vegetation V and animals X , in absence of harvesting, are given as:

$$\dot{V} = rV_t \left(1 - \frac{V_t}{K}\right) - \frac{bV_t}{V_t + a} X_t = f(V_t, X_t) \quad (1)$$

and

$$\dot{X} = \left(q \frac{bV_t}{V_t + a} - m \right) X_t = g(V_t, X_t) \quad (2)$$

respectively, where dotted variables indicate time derivatives. The parameters are interpreted as follows. r is the intrinsic growth rate of vegetation, K is the vegetation carrying capacity, b is maximum food intake per animal, a is the so-called half saturation constant that governs the curvature of the grazing function, q is the vegetation to animals biomass conversion factor and m is the animal mortality rate. Time subscripts will be dropped in the following, where no ambiguity ensues.

The first part of equation (1) is the growth function for vegetation, and implies that vegetation exhibits standard logistic growth in the absence of grazing. The second part is grazing consumption, where food intake per animal is given by $bV/(V+a)$. Therefore, the per animal consumption is an increasing and concave function of the vegetation quantity and approaches the maximum level b as food becomes abundant. Ecologists refer to this kind of grazing function as a Holling type II functional response (Holling, 1966). As indicated by equation (2), plant biomass is converted into animal biomass by a constant factor q . The mortality rate of animals is assumed fixed. Hence, the animal growth rate \dot{X}/X is independent of the number of animals, but increases with the vegetation quantity. We require that $r, K, q, b, m, a > 0$, $q, m < 1$, $a < K$ and $qb > m$, and where the last inequality implies positive animal growth when vegetation becomes sufficiently abundant. While, letting subscripts denote derivatives, we have $g_v > 0$ and $f_x < 0$ for all $V, X > 0$, the signs of g_x and f_v depend on the state of the system.

Assuming zero growth in both species, we find :

$$f(V, X) = 0 \Rightarrow X = \frac{r}{b} \left(1 - \frac{V}{K} \right) (V + a) \quad (3)$$

$$g(X, V) = 0 \Rightarrow V = a \frac{m}{qb - m} \quad (4)$$

These equations describe the isoclines of the system, for $X > 0$ and $V > 0$. Differentiating equation (3) gives $\left. \frac{dX}{dV} \right|_{\dot{V}=0} = -\frac{f_V}{f_X} = \frac{2r}{bK} \left(\frac{K-a}{2} - V \right)$. Hence, the V -isocline is bell-shaped in

(V, X) -space, with a maximum point for $V = \frac{K-a}{2}$. It intersects the V -axis at K , and the

X -axis at $\frac{ra}{b}$, see Figure 1. Moving rightwards along the V -isocline on its upward sloping

part, both the steady state animal stock and vegetation quantity increase. On the downward sloping part, there is a trade-off between the number of animals and the vegetation quantity;

that is, a negative relationship between the size of the animal stock and its rate of growth, as in a logistic growth model. The X -isocline is a vertical line crossing the horizontal axis at

$V = a \frac{m}{qb - m}$. When also including the two axes as isoclines, we find three equilibria in the

ecological model: $(0, 0)$ (semi-stable), $(0, K)$, (stable), and the interior one (X^e, V^e) with

$$X^e = \frac{r}{b} \left(1 - \frac{a}{K} \frac{m}{qb - m} \right) \left(a \frac{qb}{qb - m} \right) \quad (5)$$

$$V^e = a \frac{m}{qb - m}, \quad (6)$$

and where the superscript e indicates equilibrium values.

The parameter effects on V^e are fairly intuitive; higher animal mortality and/or lower growth rate of animals as represented by the term $qb - m$, leads to more vegetation in equilibrium, as does increasing a , so that the animals get saturated faster. Note that the parameter r , which is the intrinsic vegetation growth rate has no effect on the steady state vegetation level; an increase in vegetation productivity would be completely absorbed by a larger animal stock.

The parameter effects on X^e are more subtle however, except for the effect of r which is positive.

The interior equilibrium has interesting stability properties, which are studied by use of the Jacobian matrix of the system, linearized around the steady state. The eigenvalues, λ^+ and λ^- , are given by the roots of the equation $\lambda^2 - tr\lambda + Det = 0$, i.e.

$$\lambda^+, \lambda^- = \frac{1}{2} \left(tr \pm \sqrt{tr^2 - 4Det} \right),$$

where tr and Det denote the trace and determinant of the Jacobian, respectively. Differentiating (1) and (2) and observing that $g_x(X^e, V^e) = 0$, we find that $tr = f_v(X^e, V^e)$ and $Det = f_x(X^e, V^e)g_v(X^e, V^e)$ at the equilibrium. Since $Det > 0$ always, the expression under the root sign is negative when $tr = 0$, so that both eigenvalues are complex. Hence, in a neighbourhood of this point all equilibria are spirals, either stable or unstable. On the downward sloping part of the V -isocline we have $tr = f_v < 0$ and stable equilibria. At the upward sloping part we find unstable equilibria, see Figure 1.

[Figure 1 about here]

Choosing the animal mortality rate m as the ‘bifurcation parameter’ that shifts the X -isocline, we have that the ecological system is stable for a sufficiently high mortality rate. Starting at a cyclically stable situation on the negatively sloped part of the V -isocline and decreasing m , the system undergoes a Hopf-bifurcation when the X -isocline passes through the maximum point from the right, and a limit cycle emerges. The amplitude of the limit cycle increases as the X -isocline is shifted further to the left and may be large enough to lead to long periods of a very small animal population when the natural mortality rate is small. This type of bifurcation is a well-known phenomenon from theoretical ecology (see e.g. May, 1972).

3. Introducing harvest

When the animal stock is harvested, equation (2) is modified to

$$\dot{X} = \left(q \frac{bV}{V+a} - m \right) X - H = g(X, V) - H \quad (7)$$

where $H \geq 0$ is the harvest of animal biomass. We assume that for the region as a whole, animals cannot be imported. Hence, the harvest is nonnegative. When harvest is expressed as a fraction h of the animal stock, so that $H = hX$, a simple relationship between the harvest policy and ecological stability can be observed:

Proposition 1: If the unexploited system exhibits equilibrium cycles, global stability can be introduced by a constant harvest rate.

Proof: Replacing H with hX in eq. (7), shows that the introduction of a fixed harvest ratio is equivalent to an increase in the mortality rate from m to $m+h$. As explained in section 2, a suitably high mortality rate is sufficient for stability. \square

This result is a natural reference point for the rest of the analysis. Once an optimum is identified in the stable domain of the system, it can always be reached, and sustained, by setting the harvest ratio fixed and equal to the optimal rate. This policy is robust to fluctuations in the natural environment, and easy to implement as it only requires observations of herd size. In terms of Figure 1, we see that the fixed harvest rate has the same role as the mortality rate and shifts the X -isocline. If the equilibrium of an unexploited system is initially located to the left of the peak of the V -isocline, harvest may therefore move the system towards a stable equilibrium. The interior equilibrium is described as in eq. (5) and (6), with m replaced with $m+h$. However, and in contrast with Brekke et al (2007), who assert that a constant harvest ratio is *optimal*, with logarithmic utility in an equivalent ecological system, the optimal policy here is different as shown in section 5.

The equilibrium harvest level H^e can also be identified as a function of the equilibrium vegetation quantity. By combining equations (6) for zero animal growth and (3) we find:

$$H^e = H^e(V^e) = \left(q \frac{bV^e}{V^e+a} - m \right) \left(1 - \frac{V^e}{K} \right) (V^e + a) \frac{r}{b} \quad (8)$$

Not surprisingly, zero harvest is found when $V^e = K$ and $V^e = a \frac{m}{qb-m}$; that is, when there is no animals at all, and at the natural equilibrium (cf. eq. 1). In the first case, the harvest rate is at its maximum. In the second case, the harvest rate is zero. When $V^e \in \left(a \frac{m}{qb-m}, K \right)$, steady state harvest is positive and $H^e(V^e)$ is a concave function that reaches a peak value at $dH^e/dV^e = 0$. The vegetation level corresponding to this *maximum sustainable yield* (msy) harvest is found as $V^{msy} = \frac{1}{2} \left(a \frac{m}{qb-m} + K \right)$; that is, halfway between the natural equilibrium and the carrying capacity. See Figure 2. Notice that all the parameters of the model, except the intrinsic vegetation growth rate r , influence the *msy* vegetation level.

[Figure 2 about here]

V^{msy} is located to the right of the peak of the V -isocline because $a \frac{m}{qb-m} + K > K - a$. Because the slope of the V -isocline is described by $dX/dV|_{\dot{V}=0} = -f_V/f_X$ we then find that $f_V < 0$ at V^{msy} . In line with the results from the standard one-species model, maximizing the equilibrium harvest value for zero stock costs gives V^{msy} as the optimal solution, see discussion below.

4. The optimal harvest decision and steady state

We now study optimal harvest, assuming that the planner aims to maximize present value profit. With a domestic animal stock, there are no significant costs associated with the harvest process, but costs are instead related to stock maintenance. We assume that the herding cost $c(X)$ is a convex quadratic function of the animal stock with $c' \geq 0$, $c'' \geq 0$, $c''' = 0$, $c(0) = 0$ and $c'(0) = 0$. With p as the per animal slaughtering price (net of slaughtering costs), assumed to be fixed over time and independent of the number of animals slaughtered, the current profit is described as

$$\pi = pH - c(X). \quad (9)$$

The management goal is then formulated as

$$\begin{aligned} \max_H \int_0^{\infty} [pH - c(X)] e^{-\delta t} dt \\ \text{s.t. } \dot{X} &= g(X, V) - H \\ \dot{V} &= f(X, V) \\ X &\geq 0, V \geq 0 \\ H &\in [0, X] \end{aligned} ,$$

where $\delta \geq 0$ denotes the discount rate and X_0, V_0 are given. The current value Hamiltonian of this problem is

$$H = pH - c(X) + \lambda(g(V, X) - H) + \mu f(V, X) ,$$

where λ is the animal stock shadow price and μ is the vegetation quantity shadow price (adjoint variables). The necessary conditions, where the asterisk indicates optimal values, are:

$$H(H^*, X^*, V^*, t) = \max_{H \in [0, X]} H(H, X, V, t) \quad (10)$$

$$\dot{\lambda} = (\delta - g_X) \lambda - \mu f_X + c' \quad (11)$$

$$\dot{\mu} = (\delta - f_V) \mu - \lambda g_V . \quad (12)$$

Sufficiency is guaranteed if the Hamiltonian is concave in the states and controls jointly. This is not generally clear, because of the term $g(V)X$. A proof of sufficiency, which is not pursued here, would go along similar lines as in Asheim (2008). Transversality is satisfied because the discount rate is positive and both state variables are prevented from increasing indefinitely (see e.g Seierstad and Sydsæther, 1987).

Condition (10) entails

$$H^* = \begin{cases} 0 & p < \lambda \\ H^S & \text{if } p = \lambda \\ X & p > \lambda \quad (\text{impulse control}) \end{cases} \quad (13)$$

with so called bang-bang control, as is expected with an objective function that is linear in the control variable. H^S denotes the ‘singular’ harvest programme, which is yet to be defined from the first order conditions. The upper constraint implies that harvest is unlimited from above except by the size of the animal stock itself; and that the stock, whenever it is above the level that is consistent with singular control, will be harvested down to that level instantaneously. The solution is analyzed in two steps; we first study the optimal steady state, and then we study the transitional dynamics. The optimal steady state is defined when $\dot{X} = \dot{V} = \dot{\lambda} = \dot{\mu} = 0$. First, observe that in any optimal steady state with $X^* > 0$, the control must be singular; $H^* = H^S$. To see this, note that the harvest is limited from above only by the size of the stock, so any steady state with maximum harvest, $H^* = X^*$ necessarily means that $X^* = 0$. Moreover, setting H to its lower boundary, $H^* = 0$, cannot be optimal unless $X^* = 0$, since harvest is costless.

Imposing the interior control in the steady state, and assuming that sufficiency is satisfied, we obtain the following expression, using (11) and (12):

$$g_x = \delta + \frac{c'}{p} + \frac{g_v f_x}{f_v - \delta} \quad (14)$$

Equation (14) is a familiar optimality condition for renewable resource models, except from the last term on the right hand side, which is an interaction term capturing the ‘cost’ of grazing in terms of reduced vegetation level. To study the effect of the interaction tem, we first assume the following:

Assumption 1: In any optimal steady state, $f_v < 0$.

This implies that the optimum is found on the downward sloping part of the V -isocline. Intuition suggests that this should be the case, as any equilibrium on the upward sloping part has a counterpart on the downward sloping part with the same stock size and more harvest, which is more profitable. Moreover, the more profitable equilibrium can be reached through a

monotone increase of the harvest, by the continuity of the system. It has proven difficult to establish rigorously that this is always the case, however, except that it must hold for sufficiently low discount rates, from (14). The consequence of this assumption is that the optimum is found in the stable domain of the system. Therefore, as with the simple constant harvest ratio policy, the optimal harvest policy will also lead the system towards a stable equilibrium. Note however, that this does not say anything about the stability of the optimal harvest regime itself. As demonstrated by Wirl (1996), optimal exploitation of predator-prey systems may involve cyclical equilibrium harvest regimes. The optimal steady state is further analysed, first by setting $\delta = 0$, and then $c' = 0$.

In the case of a zero discount rate, the optimal steady state can be found as a solution to the static problem of maximizing (9) with respect to (1) and (7) when $\dot{X} = \dot{V} = 0$. In this case, the solution can be studied graphically as in figure 3. Using (14) and (3) we find a steady state relationship between H and X that acts as a constraint on the optimal solution in (X, H) -space. The optimum is then found as the tangency point between this constraint and the highest obtainable isoprofit line. The isoprofit lines start out on the second axis, and slope upwards as costs increase convexly with X . It is seen that the tangency point, in absence of discounting, must lie to the left of the top point of the sustainable harvest constraint, where the harvest is maximized (MSY).

[Figure 3 about here]

In terms of Figure 2, this means that, in the absence of discounting we must have that $V^* > V^{msy}$ when stock costs are positive. Using the concept of *ecological overgrazing*, defined by Perrings (1997) as the excess animal-to-vegetation ratio compared with what gives the highest sustainable yield, that is, $X^*/V^* - X^{msy}/V^{msy}$, this can be stated more rigorously as:

Proposition 2: With zero discounting and positive stock maintenance costs, there will be ecological undergrazing in optimum.

Proof: The LHS of (14) depends positively on V only. Increasing c' from zero, for a given animal stock size, increases the RHS of (14), which means that V^* must increase from V^{msy} .

Then $X^* < X^{msy}$ because $\left. \frac{dX}{dV} \right|_{\dot{V}=0} = -\frac{f_V}{f_X} < 0$ by Assumption 1, and the result follows. \square

Inserting from (1) and (2) in (14) and using (3) to eliminate X , we obtain the following expression for the optimal steady state vegetation level:

$$V^* = \frac{1}{4} \left(V^\infty + K \left(1 - \frac{\delta}{r} \right) + \sqrt{\left(V^\infty + K \left(1 - \frac{\delta}{r} \right) \right)^2 + 8K \frac{\delta}{r} V^\infty} \right), \quad (15)$$

$$\text{where } V^\infty = a \frac{m + \delta + \frac{c'}{p}}{qb - \left(m + \delta + \frac{c'}{p} \right)} > 0.$$

V^∞ is interpreted as follows. Rearranging the last expression gives $q \frac{bV^\infty}{V^\infty + a} - m = \delta + \frac{c'}{p}$,

which means that $g_X = \delta + \frac{c'}{p}$ when $V^* = V^\infty$, from (14). Thus, V^∞ is the vegetation level that

would be considered optimal if there was no grazing cost, that is, no feedback from the amount of predators to the abundance of prey. V^∞ increases with the discount rate and the marginal cost-price ratio, since both contribute to a smaller animal stock when the grazing term is ignored. In an open access scenario (with an infinite number of agents, hence the superscript), where each agent can be expected to behave as if ignoring the grazing cost, V^∞ is the equilibrium vegetation level resulting from individually optimizing behaviour. Condition (15) is familiar from single-species models, and can be found for instance in Conrad and Clark (1987), with a similar interpretation.

The effect of an increased discount rate is however, ambiguous as the discount rate also is included in the interaction term. The role of discounting is most easily explored if we assume

away herding costs, so that $c'/p = 0$. Setting $\delta = 0$ now gives $V^* = \frac{1}{2} \left(K + a \frac{m}{qb - m} \right) = V^{msy}$,

so that the maximum sustainable yield will be harvested, as expected. Inspection of (15) also

reveals that V^* approaches K as δ increases towards $qb\frac{K}{K+a}-m$. This means that the optimal steady state animal stock will approach zero as the discount rate approaches the maximum growth rate of the herd, in the absence of stock costs. All this fits with traditional economic reasoning, and it can also be seen that including stock costs in these two cases decreases the optimal stock size, without altering the effect of discounting. However, the general relationship between the discount rate and steady state herd size/vegetation level when the discount rate lies between the two extremes considered above is not straightforward.

Proposition 3: The effect of discounting on the optimal steady state animal stock size may be both positive and negative for small discount rates.

Differentiating (15), which can be written as $V^* = V^*(V^\infty(\delta), \delta)$ we find

$$\frac{dV^*}{d\delta} = \frac{\partial V^*}{\partial V^\infty} \frac{dV^\infty}{d\delta} + \frac{\partial V^*}{\partial \delta}, \text{ where } \frac{\partial V}{\partial V^\infty} \frac{dV^\infty}{d\delta} > 0 \text{ and } \frac{\partial V}{\partial \delta} < 0.$$

From (15), the relative magnitude of these terms depends on the specific parameter values and on the magnitude of $\frac{c'}{p}$, which is unspecified. The relationship between the discount rate and the optimal herd size

in steady state is plotted in Figure 4 for the parameter values that are used in the numerical section, except that costs are set to zero ($\gamma = 0$). As is evident from the figure, there may be a positive relationship between the discount rate and steady state herd size, for discount rates that are 'small', in this case between 0 and $\delta_1 \approx 0.06$. The exact relationship depends on the relative magnitudes of the parameters. As the steady state profit is maximized for $\delta = 0$, profits decrease when the steady state animal stock increases for $0 < \delta < \delta_1$.

[Figure 4 about here]

What more is notable, is that for a higher discount rate δ_2 , as indicated on the figure, the steady state animal stock is equal to the *msy* level; $X = X^{msy}$. As there is a one-to-one correspondence between X and V by Assumption 1, we must have $V = V^{msy}$ as well, when $\delta = \delta_2$. Then the profit is the same for both $\delta = 0$ and $\delta = \delta_2$, and for $\delta_1 < \delta < \delta_2$ steady state profit increases with the discount rate. This counter-intuitive result is known from the

theory of production with more than one capital stock (Burmeister and Hammond, 1977), and, to a lesser degree, within the resource literature (e.g. Asheim, 2008). A larger marginal rate of growth in the animal stock is required when the discount rate goes up, which is typically achieved through reducing the stock in a one-species model. But the discount rate enters the adjoint equation of both state variables, which are then both treated as productive natural capital in the optimization problem. The rate of animal growth is governed by the rate of growth in vegetation, so that it is not clear *a priori* which stock will be reduced as the discount rate increases, when there is a trade-off between the two stock across steady states.

5. The optimal approach path

The optimal approach path to the optimal steady state can be described by examining the system consisting of the two adjoint equations, together with the growth equations for vegetation and animal stock. These four differential equations together define the dynamics of the system, under singular control. As we have already identified the optimal steady state, the optimal approach path can be described by examining the Jacobian determinant, evaluated at the equilibrium. However, handling a four-dimensional determinant is in general not an easy matter. A more promising approach in the present context is to eliminate the costate variables from the system, thereby deriving an autonomous system in (V, X) space, assuming singular control. The resulting planar system can then be described using conventional graphical methods. Using equations (11) and (12), the following result is obtained:

Proposition 4: Under singular control, the dynamics of the animal stock is defined by eq. (1) and eq. (16) below:

$$g^s(X, V) = \frac{1}{c'} \left[\left(f_v - \delta - \frac{f_{xv}}{f_x} f(X, V) \right) \left(g_x - \left(\delta + \frac{c'}{p} \right) \right) - g_v f_x + g_{xv} f(X, V) \right], \quad (16)$$

where $g^s(X, V)$ denotes growth in the animal stock when harvest is singular and $f(X, V)$ is vegetation growth as defined from eq. (1)

Proof: Setting $\lambda = p$ in (11) and differentiating with respect to time - using that $f_{xx} = g_{xx} = 0$ - gives:

$$\dot{\mu} = \frac{1}{(f_x)^2} \left(\left(-p(f_{xv}(\delta - g_x) + g_{xv}f_x) - c'f_{xv} \right) \dot{V} + c''f_x \dot{X} \right)$$

Setting $\lambda = p$ in (11) once again, this time solving for μ and plugging the result into (12), we find

$$\dot{\mu} = \frac{1}{f_x} \left((\delta - f_v) \left((\delta - g_x) p + c' \right) - p g_v f_x \right)$$

Equating the two expressions for $\dot{\mu}$, recalling that $\dot{V} = f(X, V)$ from (1) and letting $\dot{X} = g^s(X, V)$ we have, after some rearranging the above expression. \square

The singular harvest rule can now be identified as function of X and V :

$$H^s = g(X, V) - g^s(X, V).$$

However, this explicit feedback rule will be quite complicated. Hence, we proceed with a qualitative analysis. Setting the bracketed expression in (16) equal to zero gives an X -isocline under singular control. Setting $f(X, V) = 0$ in this expression gives eq. (14), which is as expected because the singular X -isocline must cross the V -isocline at the optimal steady state. Using explicit expressions and rearranging, the isocline can be written as:

$$\left(\delta + 2 \frac{r}{K} \frac{V}{V+a} \left(V - \frac{K-a}{2} \right) \right) \left(q \frac{bV}{V+a} - \left(m + \delta + \frac{c'}{p} \right) \right) = q \frac{ab}{(V+a)^2} rV \left(1 - \frac{V}{K} \right) \quad (17)$$

In the special case when there are no costs associated with keeping the herd, eq. (17) is independent of X . The singular X -isocline is thus a vertical line that crosses the V -isocline and the V -axis at $V = V^*$, as given from (15) with $c'/p = 0$. Since $c'(0) = 0$ by assumption, the singular X -isocline crosses the V -axis at the same point also when costs are positive. In this case however, total differentiation of (17) and using (3) gives the slope of the singular X -isocline around the steady state:

$$\left. \frac{dX}{dV} \right|_{g^s=0} = \frac{p}{c''} \left(\frac{\left(\frac{K\delta}{r} + 2 \left(2V - \frac{K-a}{2} \right) \right)}{\left(\frac{K\delta}{r} (V+a) + 2V \left(V - \frac{K-a}{2} \right) \right)} \left(q \frac{bV}{V+a} - \left(m + \delta + \frac{c'}{p} \right) \right) + q \frac{ab}{(V+a)^2} \right) > 0$$

Hence, the singular X -isocline has a positive slope, at least in a neighbourhood of the equilibrium.

Development outside of equilibrium

To study the behaviour of the system under singular control outside of the equilibrium rigorously, we must examine the Jacobian of the new two-dimensional system. This leads to the following result:

Proposition 5: The optimal steady state is a saddle point in (V, X) -space.

Proof: See the appendix.

Equations (1) and (16) can also be used to derive the slope of the approach path around the steady state, which is shown to be positive in the appendix. There are exactly two optimal trajectories leading to the steady state under singular control, where the marginal value of the animal stock equals the market price of meat. Above this path, the shadow price of animal biomass is lower than the market price, so that the herd will be slaughtered down. Below the singular path the shadow price exceeds the market price, and harvest will be set to zero.

As indicated on the figure, there may, however, be situations where the saddle path is not feasible. This happens if parts of the singular approach path are outside the first quadrant, or when the singular approach path is too steep to be followed from the left, as the harvest is required to be nonnegative. In these cases, the premature switching principle predicts that the foresighted planner switches the control from one extreme to another ‘prematurely’ if the system will be blocked from following the singular path. Finding the exact approach path is then normally a ‘process of trial and error’ (Arrow (1968)), and hence a numerical task. We further describe the optimal approach paths in the numerical section.

6 Numerical illustration[†]

[†]The simulations were performed using the KNITRO for MATLAB solver from Ziena Optimization, with MATLAB release 2011b.

To describe the behaviour of the system numerically, the following parameter values are used: $r = 0,5$, $K = 100000$, $b = 17$, $a = 20000$, $q = 0,1$, $\delta = 0.04$ and $m = 1$. The cost function is specified as $c(X) = 0.0005 \frac{X^2}{2}$. These parameters do not constitute a realistic description of any real life grazing system, but illustrates the model with reasonable parameter values. They are calibrated to yield a maximum animal growth rate of $q \frac{bK}{K+a} - m \approx 0.4$, and the system is unstable for zero harvest. The maximum sustainable amount of animals, as a function of the steady state vegetation level, is $X \left(\frac{K-a}{2} \right) = 1059$. The number of animals that gives the highest yield is $X^{msy} = 885$, and the maximum annual harvest is $H(V^{msy}) = 262$. In the optimal steady state we have $X^* = 532$, $H^* = 196$ and $h^* = H^*/X^* = 0.37$. Figure 5 shows optimal approach paths to the steady state, from several initial states. When the animal stock is above the singular path, it will be harvested down instantaneously. Once on the optimal path, the system will follow the path, if admissible - that is, if it is in the first quadrant and not steeper than the path the system would follow with no harvest. In the opposite case, the harvest will switch to zero at some point before the singular path is reached, and stay there until the system reaches the singular path once again, this time possibly from below. As is evident from the plot, this may lead to both herd size and vegetation level overshooting the steady state, if the initial vegetation level is sufficiently low.

[Figure 5 about here]

Figure 6 shows the time development of the optimal approach paths, over the first 50 years, from the same initial states as used in Figure 5. The approach paths resulting from a constant harvest ratio, equal to the harvest rate that must hold in the optimal steady state, are also depicted. As indicated on the Figure, the approach towards the steady state is considerably slower with a constant harvest rate, especially in cases where the initial grazing pressure is far from the optimum (top left and bottom right panels).

[Figure 6 about here]

Table 1 shows the difference in total discounted profit for the first 50 years, for each initial state, between the optimal regime and the constant harvest rate regime. As the table indicates, both the two leftmost panels in Figure 5 shows situations where discounted profits are reduced by 35% by the constant harvest regime, compared to the optimal regime. The two other initial situations display smaller differences between the two harvest programs.

7. A positive stock effect

We now explore the possibility of a positive stock effect, where the stock itself contributes positively in the objective function, such that:

$$\pi = pH + b(X), \quad b' > 0, b'' < 0, b(0) = 0.$$

We assume that the stock effect is concave. This is a way of incorporating nonpecuniary returns from animal husbandry. Examples of such are many, but include the satisfaction of pursuing a traditional lifestyle, conservation of culture and tradition, and the possibility for using animals for other purposes than meat production. The possible status element present in herding communities is discussed elsewhere in the literature, as is the use of large herds as insurance against natural fluctuations in productivity (see in particular Johannesen and Skonhoft, 2011). Here we consider a marginal stock effect that is decreasing but positive for all stock sizes, implying that more is always better. More realistically perhaps; even though there might be a, say, certain status effect for small stocks, increasing costs will dominate for large stocks, so the marginal stock effect turns negative when the stock increases beyond some critical value. We are not considering this more general type of stock effect here, however.

Replacing $c' > 0$ with $-b' < 0$ in equation (14), we see that the positive stock effect contributes to decreasing the right hand side of the equation. Therefore, the vegetation level is lower and the animal stock is larger in the optimal steady state compared to the situation with a negative stock effect, as is reasonable (see also discussion in section 4). When retaining the assumption that optimal steady state is still on the negative sloping part of the V -isocline, where $f_V < 0$ (Assumption 1), the effect of the grazing cost term is as before. With respect to comparative statics, the following result can be obtained:

Proposition 6: With a positive stock effect, a price increase will lead to a smaller steady state animal stock.

This is easily seen by inspection of (14), as the coefficient of p is now negative, so that an increase in p decreases the right hand side of the equation. The intuition is that there is now a trade-off in the objective function between a large stock with low steady state harvest, and a smaller stock with higher productivity. A price increase gives harvest more weight, and hence the optimal steady state animal stock size is reduced. An increase in the discount rate will, on the other hand have the same qualitative effect as in the case of a negative stock effect, so that discounting may work either way for small discount rates. From (14), p is the only parameter whose impact on the optimal steady state depends on the stock effect. The positive stock effect itself works in the direction of an increased steady state animal stock:

Proposition 7: With zero discounting, a positive stock effect means ecological overgrazing in the steady state.

Proof: Replace $c' > 0$ with $-b' < 0$ in (14) and apply the proof of Proposition 2. \square

The optimal dynamics when the stock effect is positive can be deduced from the Jacobian determinant already examined for the case of a negative stock effect. The result in the present case is:

Proposition 8: A concave positive stock effect does not change the optimal dynamics of the system.

Proof: First recognize that c'' and $-b''$ have the same sign, and that $g_x - \left(\delta - \frac{b'}{p} \right) > 0$ must hold around the steady state, because $f_v < 0$ and hence the interaction term is positive also with a positive stock effect. Then the proof of Proposition 5 can be applied, with c' and c'' replaced by $-b'$ and $-b''$. \square

This is not unreasonable, as the concavity property of the objective function is not altered by introducing the positive stock effect. However, a convex positive stock effect ($b'' > 0$), or a

concave negative stock effect ($c'' < 0$), may lead to a cyclical approach path. This is because we then get $g_x^s < 0$ and $g_v^s > 0$, so the determinant of the Jacobian will be positive. (cfr the proof of Proposition 5). If the system is unstable at the equilibrium, the optimal steady state may then exhibit perpetual oscillations. It is also possible that parts of the optimal trajectory are not feasible, so that some combination of cyclical and bang-bang dynamics is possible, on the approach path or at the equilibrium. The existence of optimal equilibrium cycles when the objective function is non-concave is known in the resource literature, see e.g. Clark (1990), and the various alternatives for the steady state control policy in a predator-prey model similar to the one used here, are treated more thoroughly by Hoekstra and van den Bergh (2005).

8. Conclusion

This paper has analyzed the optimal management of a deterministic vegetation-herbivore ecological system, where the planner is a single profit maximizing unit. Income is assumed to be the amount of animal biomass harvested, times a constant unit price, and costs are associated with stock maintenance. This negative stock effect implies that there will always be ecological undergrazing in the optimal steady state. The effect of certain parameter changes were studied, and in particular the discount rate was found to have an ambiguous impact on the optimal steady state animal stock and vegetation level. This implies that the steady state profit may increase with discounting, for a range of discount rates. Changes in the price and cost parameter were found to have the expected impact on the steady state stocks. The optimal approach path was found to be a saddle path in state-state space. The numerical section showed that convergence was considerably faster with the optimal harvest regime, compared to a constant harvest rate regime. The difference in discounted profits between the two approaches was also shown to be substantial. Finally, the possibility of a positive, concave stock effect was discussed, and it was found that, whereas ecological overgrazing was possible and the effect of a price increase now led to a smaller animal stock, the impact of discounting and the properties of the optimal approach path were not qualitatively changed.

Naturally, the model presented here is a stylized description of an animal-vegetation system, and there are reasons to be careful when applying the theory to practical resource management. Moreover, some of the results here, such as the impact of discounting, may

seem more interesting from a theoretical than a practical point of view. The optimal harvest policy deduced here also assumes that the vegetation quantity can be carefully monitored, and if this type of monitoring is not possible, we have seen that deviations from the optimal programme might lead the system into instability. However, there are ways to implement a heuristic and robust version of the optimal regime. Once the optimal animal-to-vegetation ratio is identified, from experience and/or data and theory, keeping this ratio constant at all times will ensure that the system stays close to the optimal path, and moving towards the optimal steady state. This management rule is robust to stochastic variations in the grazing quality; whenever the quality of the pasture is subject to an exogenous shock, the animal stock is adjusted to restore the optimal animal-to-vegetation ratio as quickly as possible. We have also demonstrated that a constant harvest rate is a robust stabilizing harvest policy in this model. However, based on the optimization performed here, it seems reasonable to recommend a management regime that implies a constant animal-to-vegetation ratio, rather than a constant harvest ratio.

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Appendix

Proof of proposition 5

The saddle point property of the optimal system is ensured if the Jacobian of the linearized system is negative around the steady state. As noted in Section 4, the optimal steady state must imply singular control, and thus be characterized by eq. (17) and (1). The jacobian for this system is written as

$$J = \begin{pmatrix} f_V & f_X \\ g_V^S & g_X^S \end{pmatrix},$$

with $f_X < 0$, $f_V < 0$ (by Assumption 1),

$$g_X^S(X, V) = \frac{\delta + 2\frac{r}{K}V + a\left(\frac{V-K-a}{2}\right)}{pc''(X)} = \frac{\delta - f_V}{pc''(X)} > 0,$$

and

$$g_V^S = \frac{1}{c''(X)} \left\{ \begin{aligned} & \left[f_{VV} - \frac{f_{XV}}{f_X} f_V \right] \left[g_X - \left(\delta + \frac{c_X}{p} \right) \right] \\ & + (2f_V - \delta) g_{XV} - (g_{VV} f_X + g_V f_{XV}) \end{aligned} \right\}.$$

The sign of g_V^S is determined by the sign of the braced expression, which is the sum of three terms. The middle term is always negative, but the two other terms are more difficult to ascertain. Using (1), (2) and (3) we find that in the steady state:

$$f_{VV} - \frac{f_{XV}}{f_X} f_V = \frac{r}{K(V+a)^2} \left[a(K+a) - 2(V+a)^2 \right],$$

which is negative for $V^* \in \left[\frac{K-a}{2}, K \right]$, which holds by Assumption 1. Hence, the first term

must be negative because $g_X > \left(\delta + \frac{c_X}{p} \right)$ holds at the optimum. The third term is written as,

using (1) and (2):

$$-(g_{VV}f_X + g_Vf_{XV}) = -\frac{qab^2V}{(V+a)^4}X = f_Xg_V < 0.$$

All three terms are then negative and hence $g^S_V < 0$ around the steady state.

The determinant of the Jacobian, which is written as $|J| = f_Vg^S_X - f_Xg^S_V$, is negative given the sign of the partial derivatives deduced above. Therefore, the optimal steady state is saddle-point stable. \square

The slope of the approach path

The slope of the optimal approach path near the steady state can also be derived. Around the equilibrium, a first order approximation to $g^S(X^*, V^*)$ is given by:

$$g^S(V, X) \approx g^S_X(X - X^*) + g^S_V(V - V^*)$$

Using that $g^S = \frac{dX}{dt} \Big|_{H=H^S} = \frac{d(X - X^*)}{dt} \Big|_{H=H^S}$ since X^* is a constant, and dividing through by

$X - X^*$ gives:

$$\frac{d(X - X^*)/dt}{X - X^*} \approx g^S_X + g^S_V \frac{V - V^*}{X - X^*}.$$

We must have that $\frac{d(X - X^*)/dt}{X - X^*} < 0$ on a convergent path, so that $\frac{V - V^*}{X - X^*} > 0$, given the signs of g^S_X and g^S_V . Hence, the convergent path slopes upwards near the steady state. \square

Figures and tables

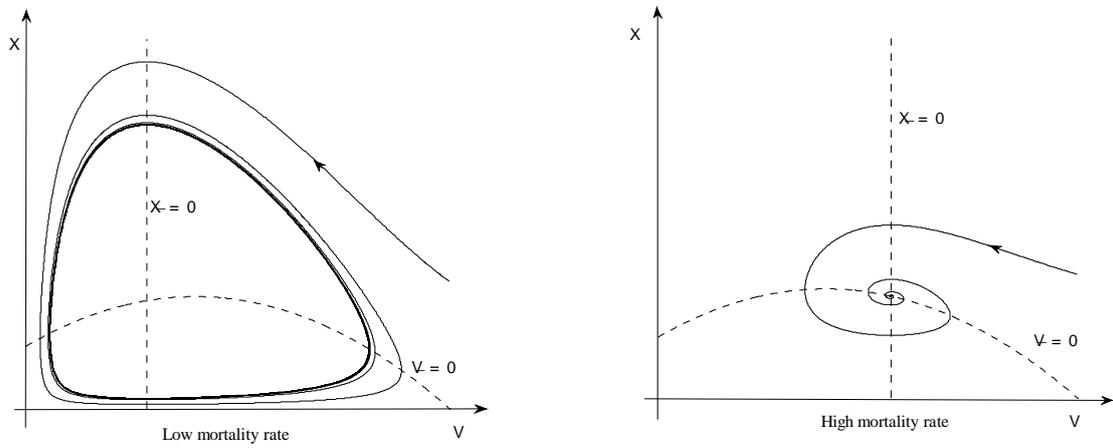


Figure 1: The unexploited system with high and low mortality rates

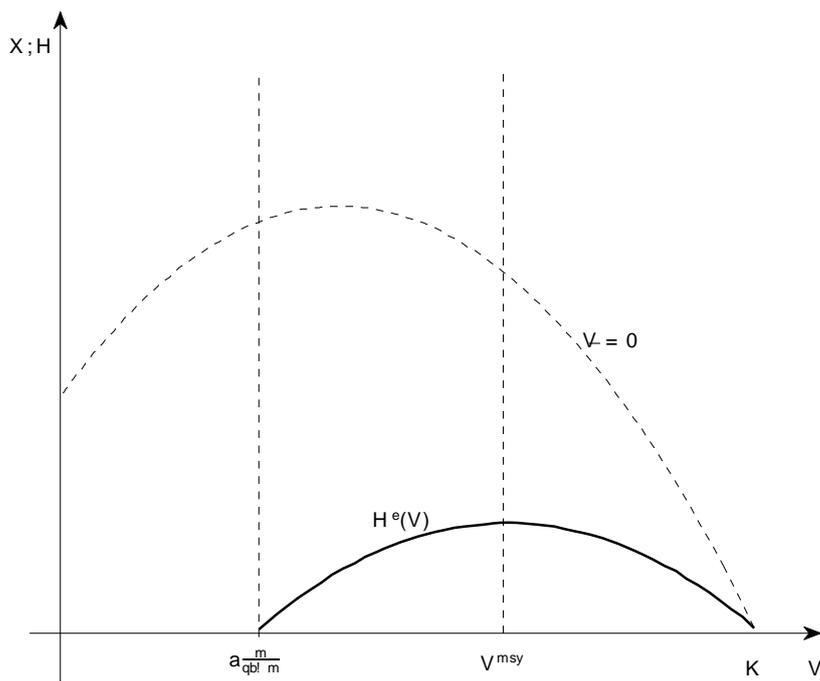


Figure 2: The sustainable harvest level as a function of V .

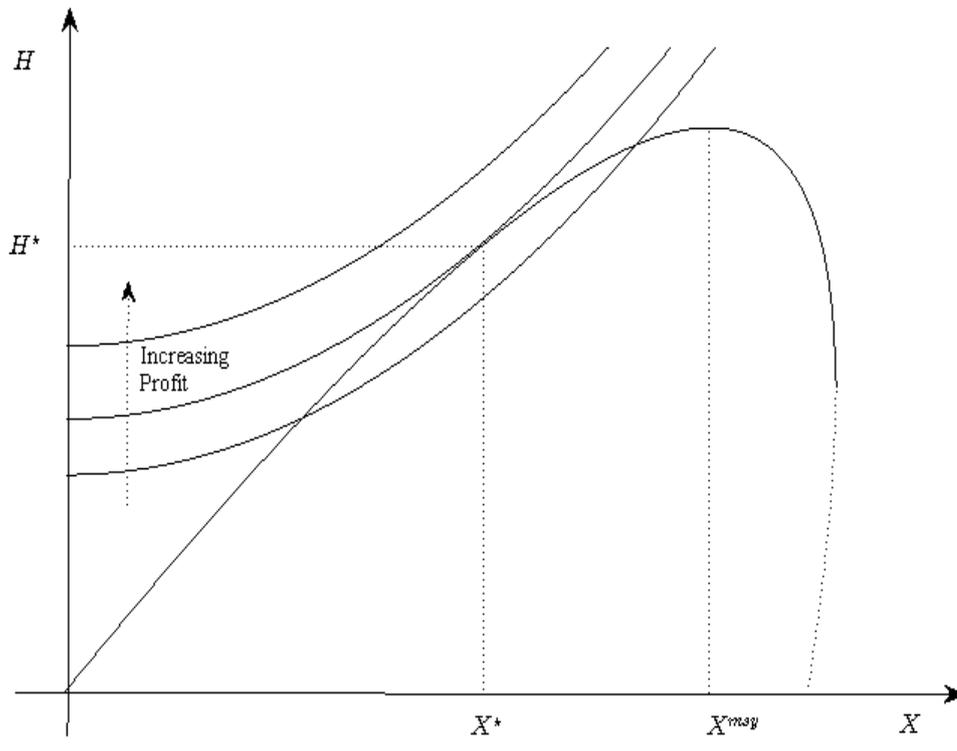


Figure 3: Correspondence between sustainable harvest and animal stock size.

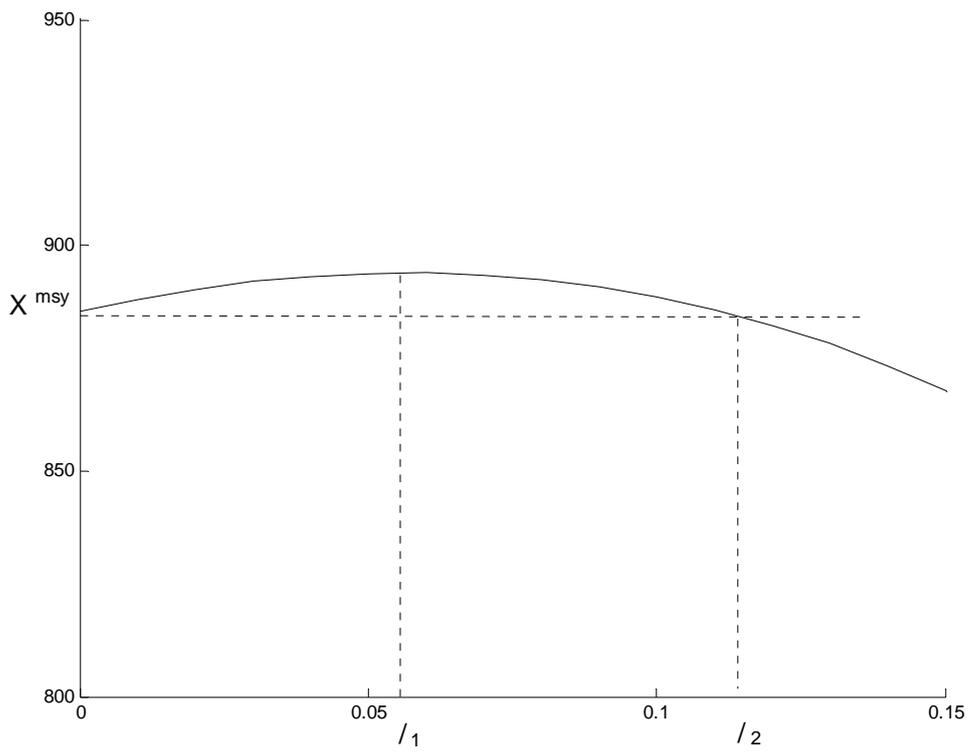


Figure 4: Steady state animal stock size as a function of the discount rate, with zero maintenance costs.

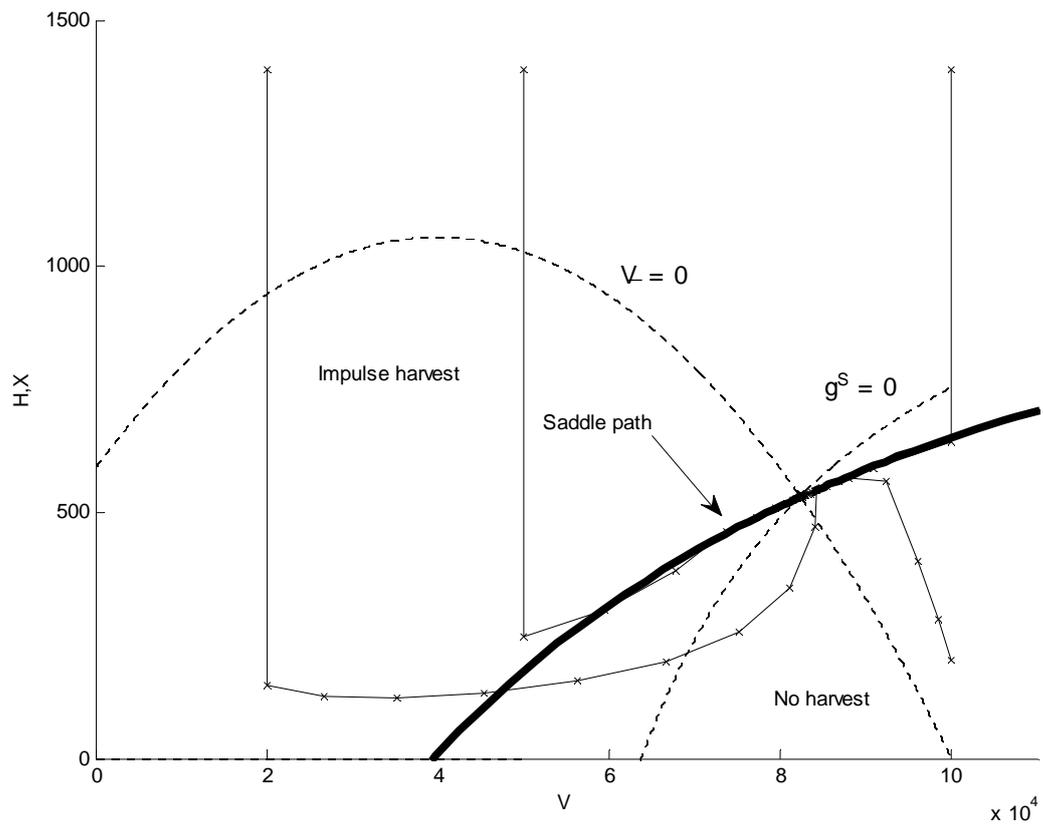


Figure 5: Optimal approach paths from four different initial states

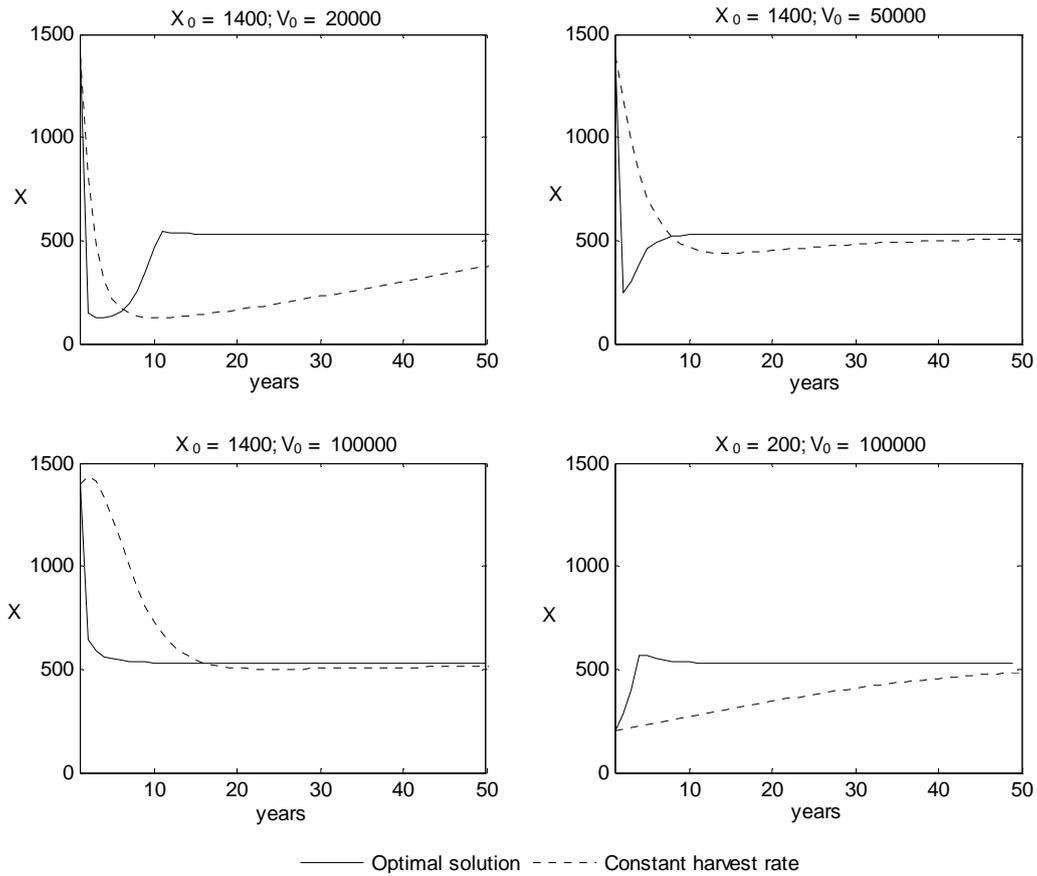


Figure 6: Time paths from four different initial states with optimal adjustment vs constant harvest rate

Table 1: Reduction in total discounted profit for the first 50 years resulting from a constant harvest rate regime, compared to the optimal program, for four initial states.

Initial state	$X_0 = 1400$ $V_0 = 20000$	$X_0 = 1400$ $V_0 = 50000$	$X_0 = 1400$ $V_0 = 100000$	$X_0 = 200$ $V_0 = 100000$
Reduction in disc. profit	-35%	-19%	-35%	-17%