Imitation Dynamics in Oligopoly Games with Heterogeneous Players

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Abstract

We investigate the role and performance of imitative behaviour in a class of quantity-setting (Cournot) games. Within a framework of evolutionary competition between rational, best-response and imitators players we found that the equilibrium stability depends on the intensity of the evolutionary pressure and on the stability of the cheapest heuristic(s). When the cheapest behavioural rule is the stable heuristic (i.e. imitation), the dynamics converge to a situation where most firms use this behavioural rule and all firms produce the Cournot-Nash equilibrium quantity. When the cheapest heuristic is unstable one (i.e. best-response), complicated endogenous fluctuations may occur.

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1 Introduction

Theocharis (1960) shows that, when firms compete on quantity using the Cournot (1838) adjustment process,[1] the Cournot model becomes unstable if the number of firms increases. In fact, with linear demand and constant marginal costs, the Cournot-Nash equilibrium loses stability and bounded but perpetual oscillations arise already for a triopoly. For more than three firms oscillations grow unbounded, but they are limited once the non-negativity price and demand constraints bind. This is a remarkable result since unbounded oscillations is not what we encounter in practice.

Whereas Theocharis focused only on the Cournot adjustment process newer research extends to models of heterogeneous expectations.[2] Hommes, Ochea

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[1] Firms that display Cournot behaviour take the *current* period's aggregate output of their competitors as a predictor for the *next* period competitors' aggregate output and best-respond to that.

[2] In models with heterogeneous expectations producers can have different heuristics to adjust their production.

and Tuinstra (2011) created a framework in which these heuristics compete in a quantity-setting. Each firm chooses a behavioural rule from a finite set of different rules, which are assumed to be commonly known. When making a choice concerning the behavioural rules, a firm takes the past performance of the rules, i.e., the past realized profit net of the cost associated with the behavioural rules to compare fitness. Both past performance and costs associated with the behavioural rules are publicly available. This implies that successful heuristics will continue to be used, while unsuccessful behavioural rules are dropped. This strategic behaviour thus causes the distribution of fractions of firms over a given set of behavioural rules to change per period.

Hommes, Ochea and Tuinstra (2011) focused on the Cournot heuristic in competition with the Nash quantity or with rational firms. Interestingly Huck, Normann and Oechssler (2002) discuss a linear Cournot oligopoly experiment with four firms. They do not find that quantities explode as the Theocharis (1960) model predicts, instead the time average quantities converge to the Cournot-Nash equilibrium quantity, although there is substantial volatility around the Cournot-Nash equilibrium quantity.

There is a growing interest, both theoretical and experimental, in the study of the performance of imitative players in various classes of games. Schipper (2009) investigates imitate-the-best players and optimizers in Cournot oligopoly and finds that in the long-run, stationary distribution of the stochastic process imitators are better off. Moreover imitation can be unbeatable if imitate-the-best heuristic is not subjected to a money pump, i.e. game is not of Rock-Scissors-Paper variety (Duersch et. al., 2012). Subsequently, Duersch et. al., 2014 show that unconditional imitation (of the tit-for-tat variety) is essentially unbeatable in class of potential games. Huck, Normann and Oechssler (2002) find that a process where participants mix between the Cournot adjustment heuristic an imitating the previous period's average quantity gives the best description of behaviour. Duersch et. al. (2009) analyse a Cournot duopoly, subjects earn on average higher profits when playing against "best-response" computers than against "imitate" computers.

Therefore we focus on competition of the imitation heuristic with the Cournot heuristic. Moreover, since classical economic theory assumes rationality, we investigate the dynamics in competition with this heuristic too. In total five models where imitators compete with Cournot and/or rational firms are investigated analytically. The framework created by Hommes, Ochea and Tuinstra (2011) will be followed in order to do the analytics. Our concern is, first of all, under what circumstances firms may want to switch between behavioural rules over time and second, once the Cournot-Nash equilibrium is reached whether all firms will keep producing the Cournot-Nash quantity or deviate.

Main findings are that, (i) in the case when Cournot firms compete with imitators that the threshold on the number of firms that changes the system from stable to unstable is 7, (ii) when rational firms compete with imitators, in the specific scenario of linear inverse demand and constant marginal cost, the system is always stable regardless of the game and behavioural parameters, (iii) in the case when rational firms, Cournot firms and imitators compete, the stability depends on the evolutionary pressure and the the stability of the cheapest heuristic(s). When the cheapest behavioural rule is stable, the dynamics converge to a situation where most firms use this behavioural rule and all firms produce the Cournot-Nash equilibrium quantity. So having more information about the market does not necessarily lead to higher profits due to information costs. In the case when the cheapest heuristic is unstable, complicated endogenous fluctuations may occur. In particular, when the evolutionary pressure is high or when the number of firms passes a certain threshold. Note that the nonlinearity causing this erratic behaviour comes from the endogenously updating of the fractions, because in our leading example the specifications were linear.

The remainder of this paper is organized as follows, in Section 2 the theoretical framework is introduced, here the quantity and population dynamics will be explained extensively. In Section 3 the dynamics will be investigated under exogenous population dynamics whereas in Section 4 the stability of the system will be investigated under endogenous population dynamics. In the fifth Section the results of section four are combined and the stability of a system where rational, Cournot and imitators compete in one economy under endogenous fraction dynamics is investigated. Finally, we conclude in Section 6.

2 Theoretical Framework

Consider a finite population of firms who are competing on the market for a certain good, each discrete-time period all producers have to decide their production plans for the next period. However, instead of simultaneously choosing the supplied quantities directly, the firms act according to behavioural rules that exactly prescribe the quantity to be supplied. Before the evolutionary model is studied a brief review of the traditional, static Cournot model will be given.

Consider a symmetric Cournot oligopoly game, where q_i denotes the quantity supplied by firm i, where $i = 1, ...n$. Next to that let $Q = \sum_{i=1}^{n} q_i$ be the aggregated production. Furthermore let $P(Q)$ denote the twice differentiable, nonnegative and non-increasing inverse demand function and let $C(q_i)$ denote the twice differentiable non-decreasing cost function, which is the same for all firms. For firm i the resulting profit function from the above described model is given by

$$\pi_i(q_i, Q_{-i}) = P(q_i + Q_{-i})q_i - C(q_i),\ i = 1, ...n \tag{1}$$

where $Q_{-i} = \sum_{j \neq i} q_j$. Assume that the profit function of a firm is strictly concave in its own output q_i. The profit maximizing strategy of firm i, taking the quantity supplied by the competitors as given, results in the well-known best-reply function for firm i, which is given by

$$q_i = R_i(Q_{-i}) = \underset{q_i}{Argmax}[P(q_i + Q_{-i})q_i - C(q_i)].$$

Due to symmetry, all firms have the same best-reply function $R(\cdot)$. Moreover, the symmetric Cournot-Nash equilibrium quantity q^* corresponds to the solution of

$$q^* = R((n-1)q^*).$$

Strict concavity of the profit function ensures that such a Cournot-Nash equilibrium exists. For simplicity assume that q^* is the unique symmetric Cournot-Nash equilibrium strategy.[3]

In this thesis focus lays on the following specification of the Cournot oligopoly game which will be called the *leading example*. This is the original specification Theocharis (1960) used, where inverse demand is linear and marginal costs are constant. The inverse demand and cost function are given by

$$P(q_i + Q_{-i}) = a - b(q_i + Q_{-i}) \text{ and } C(q_i) = cq_i, \ i = 1, ...n$$

respectively. First, in order to have a strictly concave profit function assume that $b > 0$. Furthermore, for strictly positive prices assume that $Q < \frac{a}{b}$. For these specifications of the inverse demand function and cost function the reaction function is given by

$$q_i = R(Q_{-i}) = \frac{a-c}{2b} - \frac{1}{2}Q_{-i} = q^* - \frac{1}{2}(Q_{-i} - (n-1)q^*). \qquad (2)$$

Note that if the other firms produce on average more (less) than the Cournot-Nash equilibrium quantity, firm i reacts by producing less (more) than that quantity.

Straightforward calculations show that in this case the Cournot-Nash equilibrium quantity, aggregated production, price and profit are equal to $q^* = \frac{a-c}{b(n+1)}$, $Q^* = \frac{a-c}{b}\frac{n}{n+1}$,
$P^* = \frac{a+nc}{n+1}$ and $\pi^* = \pi(q_i^*, Q_{-i}^*) = \frac{(a-c)^2}{b(n+1)^2}$.

Traditional Cournot analysis refers to a static environment. However, in a dynamic setting the reaction function introduced above can be used to study the so called Cournot-dynamics where firms best-reply to their expectations

$$q_{i,t} = R(Q^e_{-i,t}), \ i = 1, ...n$$

where $q_{i,t}$ denotes the quantity supplied by player i in period t. The symmetric Cournot-Nash equilibrium where all firms produce q^* is stable under the Cournot-dynamics if $(n-1)|R'^*)| < 1$.

Main interest is on how firm i decides to play q^* and on top of that, what does firm i believe about Q_{-i} when the production decision has to be made.

In the next Subsection the description of the quantity dynamics will be given. In Subsection 2.2 some local instability results for the general evolutionary system are discussed. In Subsection 2.3 the population dynamics will be discussed.

[3] The Cournot duopoly game may also have asymmetric Cournot-Nash equilibria, but they do not correspond to equilibria of the evolutionary game when there is a single population. For the linear-quadratic specification of the Cournot oligopoly model specified below, there can indeed be asymmetric boundary equilibria, but they do not influence the dynamics of the evolutionary model.

2.1 Production plans

In the Cournot oligopoly game the producers have to form expectations about opponents' production plans. Based on this expectation firms decide how much to produce the next period. One approach is to assume complete information, i.e. rational firms with common knowledge of rationality. This implies that firms have perfect foresight about competitors' aggregated production plan, i.e. $Q^e_{-i,t+1} = Q_{-i,t+1}$. This results in the following production plan:

$$q_{i,t+1} = R(Q_{-i,t+1}),\ i = 1, ...n$$

Alternatively one may consider rules that require less information, for example $Q^e_{-i,t+1} = Q_{-i,t}$. This results in the following production plan:

$$q_{i,t+1} = R(Q_{-i,t}),\ i = 1, ...n \tag{3}$$

where firms expect that aggregated production in the next period equals current aggregated production. This is the so called Cournot adjustment heuristic.

It is a broadly supported idea that not all producers best-reply to their expectations. Experiments (Huck 2002) show that people often imitate others' behaviour. A heuristic that possibly seizes this production plan is the so called imitation-heuristic. Imitators belief that "everyone else can't be wrong" and will therefore produce the average of the other players' production in the next period, i.e.

$$q_{i,t+1} = \frac{Q_{-i,t}}{n-1},\ i = 1, ..., n. \tag{4}$$

Finally, Bosch-Domènech and Vriend (2003) test the importance of models of behaviour characterised by imitation of successful behaviour, that is to imitate the quantity which the firm with the highest profit in the current period produced, i.e.

$$q_{i,t+1} = q_{j,t},\ i = 1, ...n,\ \text{where } \Pi_{j,t} = Max\{\Pi_{1,t}, ..., \Pi_{k,t}\}.$$

They find that the players do not rely more on imitation of successful behaviour in more demanding environments and explain the different output decisions as predominantly relate to a general disorientation of the players, and more specifically to a significant decrease of best responses.

In the next subsection we will investigate the dynamics under expectation rule (3) and (4) in greater detail.

2.2 Instability threshold

2.2.1 Cournot adjustment heuristic

If all firms use the Cournot adjustment heuristic (3), quantities evolve according to the following system of n first order difference equations

$$\begin{aligned} q_{1,t+1} &= R(q_{2,t} + q_{3,t} + \ldots + q_{n,t}), \\ q_{2,t+1} &= R(q_{1,t} + q_{3,t} + \ldots + q_{n,t}), \\ &= \\ q_{n,t+1} &= R(q_{2,t} + q_{3,t} + \ldots + q_{n,t}). \end{aligned} \tag{5}$$

Local stability of the Cournot-Nash equilibrium depends on the eigenvalues of the Jacobian matrix J of the system of equations (5), evaluated at that Cournot-Nash equilibrium q^*. This Jacobian matrix is given by

$$J|_{q^*} = \begin{pmatrix} 0 & R'(Q^*_{-1}) & \cdots & R'(Q^*_{-1}) \\ R'(Q^*_{-2}) & 0 & & \vdots \\ \vdots & & \ddots & R'\left(Q^*_{-(n-1)}\right) \\ R'(Q^*_{-n}) & \cdots & R'(Q^*_{-n}) & 0 \end{pmatrix}. \tag{6}$$

Firms do not respond to their own previous production, therefore all diagonal elements are equal to zero. All off-diagonal elements in row i are equal to $R'(Q^*_{-i})$, since individual production levels only enter through aggregate production of the other firms. Moreover, at the symmetric Cournot-Nash equilibrium we have $Q^*_{-i} = (n-1)q^*$ for $i = 1, ..., n$, thus all off-diagonal elements of (6) are equal to $R'^*)$. The Jacobian matrix (6) thus has $n-1$ eigenvalues equal to $-R'^*)$ and one eigenvalue equal to $(n-1)R'^*)$, which is the largest in absolute value. From this it follows directly that the symmetric Cournot-Nash equilibrium is stable whenever

$$\lambda(n) \equiv (n-1)|R'^*)| < 1, \tag{7}$$

where $\lambda(n)$ is defined as the largest eigenvalue of the Jacobian, evaluated at the equilibrium.

Leading example. From equation (2) it can easily be seen that $R'(Q^*_{-i}) = -\frac{1}{2}$, meaning that if others' aggregated output increases by one unit, the Cournot-Nash firms decrease their output by $\frac{1}{2}$ units. From stability condition (7) it follows that the Cournot-Nash equilibrium is stable for this specification only when $n = 2$ and unstable when $n > 3$ (and neutrally stable, resulting in bounded oscillations, for $n = 3$). The reason for this instability is 'overshooting': if aggregated output is above (below) the Cournot-Nash equilibrium quantity, firms react by reducing (increasing) their output. For $n > 3$ this aggregated reduction (increase) in output is so large that the resulting deviation of aggregated output from the equilibrium quantity is larger in the next period than in the current, and so on.

2.2.2 Imitation heuristic

If all firms use the imitation heuristic (4), quantities evolve according to the following system of n equations

$$\begin{aligned} q_{1,t+1} &= \frac{Q_{-1,t}}{n-1}, \\ q_{2,t+1} &= \frac{Q_{-2,t}}{n-1}, \\ &= \\ q_{n,t+1} &= \frac{Q_{-n,t}}{n-1}. \end{aligned} \tag{8}$$

Local stability of the Cournot-Nash equilibrium with only imitation firms depends on the eigenvalues of the Jacobian matrix of the system of equations (8) evaluated at that Cournot-Nash equilibrium q^*. This Jacobian matrix is given by

$$J|_{q^*} = \begin{pmatrix} 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & 0 & & \vdots \\ \vdots & & \ddots & \frac{1}{n-1} \\ \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 \end{pmatrix}. \tag{9}$$

Imitators only respond to other firms' production and do not respond to their own production, therefore all diagonal elements are equal to zero. If one competitor increases *current* production by one unit, an imitator will increase *next* production with $\frac{1}{n-1}$ unit, therefore all off-diagonal elements are equal to $\frac{1}{n-1}$. The Jacobian matrix (6) thus has $n-1$ eigenvalues equal to $-\frac{1}{n-1}$ and one eigenvalue equal to $(n-1)\frac{1}{n-1} = 1$ which is the largest in absolute value. Therefore it follows immediately that the Cournot-Nash equilibrium is neutrally stable independent of n and system structure (price and cost function). The reason for this is that if one producer changes his production plan the economy will stabilize to a new equilibrium unequal to q^* and will remain at this new equilibrium until one producer deviates again. In fact this system has infinitely many neutrally stable equilibria, namely if $q_i = q^\dagger\ \forall i$, the system is neutrally stable for all $q^\dagger$.

2.3 Population dynamics

In the previous sections it is explained how the supplied quantities evolve over time under the Cournot and the imitation heuristic. In this section it will be explained how the population fractions evolve over time. Let us first introduce the vector η_t which has entries equal to $\eta_{k,t}$, which is the fraction of the population that uses heuristic k at time t. Thus for every time t, η_t denotes the K-dimensional vector of fractions for each strategy/heuristic and belongs to the K-dimensional simplex $\Delta^K = \{\eta_t \in \mathbb{R}^K : \sum_{k=1}^K \eta_{k,t} = 1; 0 \leq \eta_{k,t} \leq 1\ \forall k\}$. We will now describe how the fractions $\eta_{k,t}$ evolve over time. It is assumed that the choice of a behavioural rule is based on its past performance, capturing the idea that more successful rules will be used more frequently.

Evolutionary game theory deals with games played within a (large) population over a long time horizon. Its main ingredients are its underlying game, in this thesis the Cournot one-shot game, and the evolutionary dynamic class which defines a dynamical system on the state of the population. The evolutionary dynamical system depends on current fractions η_t and current fitness U_t. In general, such an evolutionary dynamic in discrete time, describing how the population fractions evolve, is given by

$$\eta_{k,t+1} = K(U_t, \eta_t) \tag{10}$$

with $U_t = (U_{1,t}, ..., U_{K,t})'$ the vector of average utilities and $\eta_t = (\eta_{1,t}, ..., \eta_{K,t})'$ the factor of fractions. To make sure that the population dynamics is well-behaved in terms of dynamic implications we assume that $K(\cdot, \cdot)$ is continuous, nondecreasing in $U_{k,t}$, and such that the population state remains in the K-dimensional unit simplex Δ^K. In the next Subsection leading class of population dynamics will be explained in detail, the Logit evolutionary dynamics.

2.3.1 Discrete choice models - the Logit evolutionary dynamics

The Logit evolutionary dynamic is treated extensively in Brock and Hommes (1997). This Section contains a brief discussion.

In order to update the fractions we assume that average utility of all heuristics is publicly observable. Suppose that the observed average utility associated behavioural rule H_k takes the form

$$\tilde{U_k} = U_k + \frac{1}{\beta}\epsilon_k,$$

where ϵ_k's are IID. This captures the idea of bounded rationality since individuals do not necessarily select the rule that yields the highest utility. The parameter β represents the evolutionary pressure. Notice that in the extreme case where $\beta = 0$ we have completely random behaviour: the noise is so large that observed average utility is equal for all behavioural rules. Each behavioural rule is thus chosen with equal probability: $\eta_{k,t} = \frac{1}{K}\ \forall k$. In the other extreme case, when $\beta \to \infty$ obscures and everybody switches to the most profitable strategy each period. If the noise terms ϵ_k's are distributed according to the extreme value distribution the evolutionary fraction dynamic results in the so-called multinomial Logit evolutionary dynamic, the following updating dynamic is given by

$$\eta_{k,t+1} = \frac{e^{\beta U_{k,t}}}{\sum_{j=1}^{K} e^{\beta U_{j,t}}},\ k = 1, ..., K. \tag{11}$$

The equilibrium fractions are given by

$$\eta_{k,t+1} = \frac{e^{\beta(\Pi^* - T_k)}}{\sum_{j=1}^{K} e^{\beta(\Pi^* - T_j)}},\ k = 1, ..., K \tag{12}$$

In case of equal costs of the heuristics, equilibrium fractions are thus given by $\eta_k^* = \frac{1}{K}\ \forall k$, since production is equal and thus profits are equal. Note that the population dynamics remains in the interior of the unit simplex for finite β. This implies that in each time period all behaviour rules are present in the population and no behavioural rule will ever vanish (this is the so-called *no-extinction* condition). Furthermore, no new behavioural rules emerge from this model (this is the so-called *no-creation* condition).

In the *leading examples* we will focus on the Logit evolutionary dynamics. First of all because this dynamic is also used in Hommes, Ochea and Tuinstra (2011) and therefore creates the possibility to make a good comparison and furthermore because the Logit evolutionary dynamic has by definition nice regularity/continuity conditions ($0 \leq \eta_k \leq 1$).

3 Heterogeneity in behaviour in Cournot oligopolies

In this Section we study the Cournot game and introduce heterogeneity in production plans. In this Section we focus on competition between two heuristics. We relax this in Section five, where we study the competition between rational, Cournot and imitation firms. First we study competition between the Cournot and the imitation firms, with this as an example, two theories will be presented on how to model this heterogeneity in production. In the first theory the firms select their heuristic that completely describes how much to supply in the next period. They select heuristic k with probability η_k. In the second theory n firms are randomly picked from a large population of firms in which a fraction η_k plays according to strategy k. Main difference is that the firms observe under the second theory more outcomes and thus under the law of large numbers lets the production plans within a heuristic converge whereas in the first theory all firms (even the firms using the same heuristic) have different production plans, making the dynamics analytically untractable. After this extensive study of competition between Cournot firms and imitators, we introduce another model where rational firms compete with imitation firms. Since the dynamics are only tractable under theory 2, we will focus on this theory when studying this model. The assumption of fixed η for each period will be relaxed in section 4.

3.1 Cournot vs. Imitation firms

3.1.1 Theory 2: A large population game

In order to facilitate studying the aggregate behaviour of a heterogeneous set of interacting quantity-setting-heuristics we study the Cournot model as a population game. Consider a large population of firms from which in each period groups of n firms are sampled randomly and matched to play the one-shot n-player Cournot game. We assume that a fixed fraction of η of the large population of firms uses the Cournot heuristic and the others use the imitation heuristic. After each one-shot Cournot game, the random matching procedure is repeated, leading to new combinations types of firms. The distribution of possible samples follows a binomial distribution with parameters n, and η. Below the example Cournot vs. Imitation firms will be discussed again but now under theory 2 of random matching.

Suppose that a fraction of η of the population of the firms uses the Cournot heuristic and observes the population-wide average quantity $\bar{q}_t$ and best responds to it, $q^C_{t+1} = R((n-1)\bar{q}_t)$, where q^C_t is the quantity produced by each Cournot firm in period t. Consequently a fraction of η firms of the large population makes use of the the imitation heuristic. Making use of the law of large numbers, the average quantity played in period t can be expressed as

$$\bar{q}_t = \eta q^C_t + (1-\eta)q^I_t.$$

Remember that imitation firms produce in the next period the by the other firms average produced quantity in the current period $q^I_{i,t+1} = \frac{Q_{-i,t}}{n-1}$. Again by a law of large numbers we obtain $\frac{Q_{-i,t}}{n-1} \to \bar{q}_t$when n $\to \infty$. Therefore we obtain the following quantity dynamics

$$\begin{aligned} q_{t+1}^C &= R((n-1)(\eta q_t^C + (1-\eta) q_t^I)) \\ q_{t+1}^I &= \eta q_t^C + (1-\eta) q_t^I. \end{aligned} \tag{13}$$

Note that this is a 2-dimensional dynamical system which dimension cannot be reduced. Furthermore the Cournot-Nash equilibrium is not the unique equilibrium of the imitation rule, in fact all quantities are. The Cournot-Nash equilibrium is, however, still the unique equilibrium quantity of the complete dynamical system.

Proposition 1 *The Cournot-Nash equilibrium,where all firms produce the Cournot-Nash quantity (q^*, q^*), is a locally stable fixed point for the model with exogenous fractions of Cournot and imitation firms if and only if*

$$|1 - \eta + \eta(n-1)(R'^*)| < 1. \tag{14}$$

Proof. It can easily be shown that the Jacobian matrix, evaluated at the Cournot-Nash equilibrium (q^*, q^*), is given by

$$\begin{pmatrix} (n-1)\eta R'^*) & (n-1)(1-\eta)R'^*) \\ \eta & 1-\eta \\ & \end{pmatrix}. \tag{15}$$

The corresponding eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1 - \eta + \eta(n-1)(R'^*)$. Here λ_2 is the largest eigenvalue in absolute value. Thus the system is stable if $|\lambda_2| < 1$, this is the condition stated in the proposition.
Leading example. Here $R'^*) = -\frac{1}{2}$ substituting this in equation (14) gives, after some simplification

$$n < \frac{4-\eta}{\eta}. \tag{16}$$

Meaning that an economy with as much Cournot firms as imitators($\eta = \frac{1}{2}$) is stable if $n < 7$. Next to that as found earlier, an economy with only cournot firms ($\eta = 1$) is stable if $n < 3$. Furthermore, an economy where close to all firms use the imitation heuristic, but some Cournot firms exist (η close to zero), the economy is always stable.

3.2 Rational vs. Imitation firms

In this section we focus on the dynamics when there is competition between rational and imitation firms. Remember that we will model this heterogeneity under theory 2 since this makes the dynamics analytically tractable. We set the fraction of rational firms equal to η. A fully rational firm is assumed to know the fraction of imitation firms. Moreover, it knows exactly how much all firms will produce. However, we assume that it does not know the composition of firms in its market (or has to make a production decision before observing this). The rational quantity dynamics therefore have the following structure

$$q^R = \underset{q_i}{Argmax} E[P(q_i + Q_{-i})q_i - C(q_i)].$$

It forms expectations over all possible mixtures of heuristics resulting from randomly drawing $n-1$ other players from a large population, of which each with chance η is a rational firm too, and with chance $1-\eta$ is an imitator. Rational firm i therefore chooses quantity q_i such that his objective function, its own expected utility

$$U_t^R(q_{i,t}|q_t^R, q_t^I, \eta) = \sum_{k=0}^{n-1} \binom{n-1}{k} \eta^k (1-\eta)^{n-1-k} [P((n-1-k)q_t^I + kq_t^R + q_{t,i})q_{t,i} - C(q_{t,i})], \tag{17}$$

is maximized given the production of the other players and the population fractions. Here q_t^R is the symmetric output level of all of the other rational firms in period t, and q_t^I is the output level of all of the imitation firms. The first order condition for an optimum is characterized by equality between marginal cost an *expected* marginal revenue. Typically, marginal revenue in the realized market will differ from marginal costs.

Given the value of q_t^I and the fraction η, all rational firms coordinate on the same output level q_t^R. This gives the first order condition

$$\frac{\delta U_t^R(q_{i,t}|q_t^R, q_t^I, \eta)}{\delta q_{i,t}} = 0,$$

which equals to:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \eta^k (1-\eta)^{n-1-k} \times$$
$$[P((n-1-k)q_t^I + (k+1)q_t^R) + q_t^R P'((n-1-k)q_t^I + (k+1)q_t^R) - C'(q_t^R)] = 0. \tag{18}$$

Let the solution to equation (18) be given by $q_t^R = H^R(q_t^I, \eta)$, the full system of equations is thus given by

$$\begin{aligned} q_{t+1}^R &= H^R(q_{t+1}^I, \eta) = H^R(\eta q_t^R + (1-\eta)q_t^I, \eta) \\ q_{t+1}^I &= \eta q_t^R + (1-\eta)q_t^I. \end{aligned} \tag{19}$$

It is easily checked that if the imitators play the Cournot-Nash equilibrium quantity q^*, or if all firms are rational, the rational firms will play the Cournot-Nash equilibrium quantity, that is $H^R(q^*, \eta) = q^*$, for all η and $H^R(q^I, 1) = q^*$ for all q^I. Moreover, if a rational firm is certain it will only meet imitation firms (that is $\eta = 0$), it plays a best response to the currently average played quantity, that is $H^R(q_t^I, 0) = R((n-1)q_t^I)$, for all q_t^I. In the remainder we will denote the partial derivatives of $H^R(q, \eta)$ with respect to q and η by $H_q^R(q, \eta)$ and $H_\eta^R(q, \eta)$ respectively.

Proposition 2 *The Cournot-Nash equilibrium, where all firms produce the Cournot-Nash quantity (q^*, q^*), is a locally stable fixed point for the model with exogenous fractions of rational and imitation firms if and only if*

$$|\eta H_q(q*, \eta) + 1 - \eta| < 1 \tag{20}$$

Proof. In order to determine the local stability of the equilibrium (q^*, q^*) where all firms produce the Cournot-Nash quantity, we need to determine the eigenvalues of the Jacobian matrix of system (19), evaluated at the equilibrium. It can be shown that this Jacobian matrix is given by

$$J|_{q^*,q^*} = \begin{pmatrix} \eta H_q(q^*, \eta) & (1-\eta) H_q(q^*, \eta) \\ \eta & 1-\eta \end{pmatrix}, \tag{21}$$

which has eigenvalues $\lambda_1 = \eta H_q(q^*, \eta) + 1 - \eta$ and $\lambda_2 = 0$. Consequently the system is locally stable when $|\lambda_1| < 1$, this is exactly the condition stated.
Leading example. In the leading example the implicit function defining q_t^R (Eq. (18)) when using that

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \eta^k (1-\eta)^{n-1-k} = 1 \text{ and } \sum_{k=0}^{n-1} \binom{n-1}{k} \eta^k (1-\eta)^{n-1-k} k = (n-1)\eta$$

boils down to

$$q_{t+1}^R = H^R(q_{t+1}^I, \eta) = \frac{a-c}{b(2+(n-1)\eta)} - \frac{(n-1)(1-\eta)}{2+(n-1)\eta}(\eta q_t^R + (1-\eta) q_t^I).$$

The system of equations for the leading example is given by

$$\begin{aligned} q_{t+1}^R &= H^R(q_{t+1}^I, \eta) = \frac{a-c}{b(2+(n-1)\eta)} - \frac{(n-1)(1-\eta)}{2+(n-1)\eta}(\eta q_t^R + (1-\eta) q_t^I) \\ q_{t+1}^I &= \eta q_t^R + (1-\eta) q_t^I \end{aligned} \tag{22}$$

The eigenvalues of this system are given by $\lambda_1 = 0$ and $\lambda_2 = 1 - \eta - \frac{(n-1)(1-\eta)\eta}{2+(n-1)\eta}$, thus the system is stable if $|\lambda_2| < 1$. Since $0 \leq \frac{(n-1)(1-\eta)\eta}{2+(n-1)\eta} < 1$, this stability condition always holds and the economy is always stable in the linear specification.

In Figure 1 this is graphically shown.

4 Evolutionary competition between two heuristics

In this Section we develop an evolutionary version of the model outlined in Section 3, i.e. relaxing the assumption that η is fixed. As before in ever period t, n firms play the n-player Cournot game. We now assume that the fractions of firms using a heuristic η evolves over time according to a general monotone selection dynamic, capturing the idea that heuristics that perform relatively better are more likely to spread through the population as explained in Section 2.3, Eq. (10), here it is explained that future fractions depend on current fractions and current fitness.

Under the assumption of random interactions, the fitness of heuristic k is determined by averaging the payoffs from from each interaction with weights given by the chance of that specific state minus the information cost of using the heuristic. Denoting with Π_t the expected payoff vector in period t, its entries - individual payoff or fitness in biological terms - of strategy 1 is given by:

$$\Pi_{1,t} = F(q_{1,t}, q_{2,t}, \eta_t) =$$
$$\sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} \eta_t^k (1-\eta_t)^{n-1-k} P((k+1)q_{1,t} + (n-1-k)q_{2,t})q_{1,t} - C(q_{1,t}), \tag{23}$$

and with expected profits for heuristic 2 given by $\Pi_2 = F(q_2, q_1, 1-\eta)$. If the population of firms and the number of groups of n firms drawn from that population are large enough, average profits will be approximated well by these expected profits, which we will use therefore as a proxy for average profits from now on.

There might be a substantial difference in sophistication between different heuristics. As a consequence some heuristics may require more information or effort to implement than others. Therefore we allow for the possibility that heuristics involve *information cost* $C_k \geq 0$, that may differ across heuristics. Fitness of a heuristic is then given by the average profits generated in the game minus the information costs, $U_k = \Pi_k - C_k$. We only use the realized profit to determine the fitness measure of a behavioural rule. The fitness measure can be generalized by weighting the utility of the past M periods, according to Tuinstra (1999) this yields similar results. We assume that the above fitness measures U_k are publicly observable.

Having the fitness measure we are ready to introduce the population dynamics. Let the fraction of firms using the first heuristic be given by η in period t. This fraction evolves endogenously according to an evolutionary dynamic which is an increasing function in the difference between the current fitness of the two heuristics and current fraction, that is

$$\eta_{t+1} = K(U_{1,t} - U_{2,t}) = K(\Delta U_{1,t}).$$

The map $K : \mathbb{R} \to [0, 1]$ is a continuously differentiable, monotonically increasing function with $K(0) = \frac{1}{2}$, $K(x) + K(-x) = 1$, meaning that it is symmetric around $x = 0$, $\lim_{x\to-\infty} K(x) = 0$ and $\lim_{x\to\infty} K(x) = 1$

In the following two sections we will derive two dynamical versions of the two models discussed in Section 3 and investigate their stability. First we investigate the stability of the Cournot-Nash equilibrium for the model with endogenous fractions of Cournot and imitation firms and second we investigate the stability of the Cournot-Nash equilibrium for the model with endogenous fractions of rational and imitation firms.

4.1 Cournot versus Imitation firms

The dynamics in this section consists of three equations, two equations describing the quantity dynamics: the production of the Cournot firms and the production of the imitation firms. Next to that we need one equation to describe the dynamics of the population fraction. The population and quantity dynamics look like the following system of three equations:

$$\begin{aligned} q^C_{t+1} &= R((n-1)(\eta_t q^C_t + (1-\eta_t) q^I_t)), \\ q^I_{t+1} &= \eta_t q^C_t + (1-\eta_t) q^I_t \\ \eta_{t+1} &= K(\Delta U_t), \end{aligned} \tag{24}$$

where $\Delta U_t = U_{C,t} - U_{I,t}$. Note that this is a 3-dimensional dynamical system which dimensions cannot be reduced. Furthermore, the Cournot-Nash equilibrium quantity q^* is the unique equilibrium quantity of the complete dynamical system. Let η^* be the unique equilibrium fraction such that $\eta^* = K(-C)$. Without specializing the population dynamics $K(\cdot)$ we have the result as stated in the proposition below.

Proposition 3 *The Cournot-Nash equilibrium (q^*, q^*, η^*) is a locally stable fixed point for the model with endogenous fractions of Cournot and imitators where all firms produce the Cournot-Nash quantity, firms if and only if*

$$\eta^* R((n-1)q^*)(n-1) - \eta^* > -2. \tag{25}$$

Proof. It can easily be shown that the Jacobian matrix of system 24, evaluated at the equilibrium (q^*, q^*, η^*) is given by

$$J|_{q^*,q^*,\eta^*} = \begin{pmatrix} (n-1)\eta^* R'^*) & (n-1)(1-\eta^*)R'^*) & 0 \\ \eta^* & 1-\eta^* & 0 \\ J_{31} & J_{32} & \frac{\delta K(\Delta U_t)}{\delta_{\eta_t}}\Big|_{q^*,q^*,\eta^*} \end{pmatrix}. \tag{26}$$

The eigenvalues of this Jacobian matrix are, independently of J_{31} and J_{32} given by

$$\lambda_1 = \eta^* R((n-1)q^*)(n-1) + \eta^* - 1, \;\; \lambda_2 = \frac{\delta K(\Delta U_t)}{\delta_{\eta_t}}\Bigg|_{q^*,q^*,\eta^*} \text{ and } \lambda_3 = 0. \tag{27}$$

To our best knowledge of possible population dynamics $\frac{\delta K(\Delta U_t)}{\delta_{\eta_t}}$ is positive but smaller than 1. This holds for all population dynamics discussed in Section 2.3. Therefore, for the system to be stable we need

$$\eta^* R((n-1)q^*)(n-1) - \eta^* > -2,$$

which is exactly the condition stated in the proposition.

Note that this is the same condition we derived in Section 3.1.1 where we fixed η.

Leading example. In the equilibrium, when all firms produce the same quantity, profits are equal and therefore the equilibrium fraction simplifies to $\eta^* = K(-C)$. The equilibrium quantities are given by q^*. Here $R'^*) = -\frac{1}{2}$, filling this in equation (25) gives the stability condition for the leading example. Thus the equilibrium (q^*, q^*, η^*) is stable when $n < \frac{4-\eta^*}{\eta^*}$.

In Figure 2 the model is simulated under Logit-dynamics with intensity of choice parameter β, see Brock and Hommes (1997). Panel (a) depicts a period-doubling route to chaotic quantity dynamics as the number of firms n increases. The first period-doubling bifurcation is for $n = 7$ as calculated analytically. Panel (b) displays oscillating time series of produced quantity by the Cournot and imitation firms and the equilibrium quantity fraction q^*. As one can see the Cournot quantities are fluctuating more than the imitation quantities. The stabilizing effect of the imitation firms is here clearly visible, when Cournot firms produce more (less) then the Cournot-Nash equilibrium quantity, the imitation firms produce less (more) than the Cournot-Nash equilibrium quantity and therefore decrease the aggregated deviation from the equilibrium. Panel (c) displays the resulting Cournot profit differential $\Pi^C - \Pi^I$. Panel (d) displays the resulting oscillating time series of the Cournot and imitation fractions. In Panel (e) a phase portrait is shown for the Cournot heuristic whereas in Panel (f) a phase portrait for the imitation heuristic is shown. In Panel (g) the largest Lyapunov exponent for an increasing number of firms is shown. Game and behavioural parameters are equal set to: $n = 10$, $a = 17$, $b = 1$, $c = 1$, $C^C = 0$, $C^I = 0$, $\beta = 0.05$. Initial conditions are set equal to: $q_0^C = 0.8$, $q_0^I = 0.8$, $\eta_0 = 0.5$ When the evolutionary pressure increases, the system evolves to an equilibrium different from the Cournot-Nash equilibrium where the imitation firms produce more than the Cournot-Nash equilibrium whereas the Cournot firms produce less. Imitation profits are therefore much higher and as a consequence the complete population switches to the imitation heuristic.

The bifurcation diagram is re-plotted in Fig. 3 under the same game and behavioural parameters and initial conditions, the only difference is that now $\beta = 3$.

When $1.7 < n < 2.8$ the imitation firms produce *more* then the Cournot-Nash equilibrium quantity while the Cournot firms produce *less*. This results in higher profits for the imitators and therefore the complete populations switches to imitators ($\eta = 0$). When $2.8 \leq n \leq 3.2$ all firms produce the Cournot-Nash equilibrium quantity again, therefore profits and thus fractions are equal. When $n > 3.2$ The imitation firms produce again *more* then the equilibrium quantity while the Cournot firms produce *less*, exept when n is close to 3.65, then all firms produce the Cournot-Nash equilibrium quantity. Finally, when $n > 5.6$ the imitation firms produce so much that the Cournot firms decide to produce nothing ($q^C = 0$).

4.2 Rational vs. Imitation firms

As in the previous Section we need a 3-dimensional system to describe the dynamics of the model. The rational firms produce each period such that their *expected* profit is maximized whereas an imitator produces in the next period the currently average played quantity.

The rational quantity dynamics therefore have the following structure

$$q_t^R = \underset{q_i}{Argmax} E[P(q_{i,t} + Q_{-i,t})q_i - C(q_{i,t})].$$

It forms expectations over all possible mixtures of heuristics resulting from randomly drawing $n-1$ other players from a large population, of which each with chance η_t is a rational firm too, and with chance $1-\eta_t$ is a imitator. Rational firm i therefore chooses quantity q_i such that his objective function, its own expected utility

$$U_t^R(q_{i,t}|q_t^R, q_t^I, \eta_t) = \sum_{k=0}^{n-1} \binom{n-1}{k} \eta_t^k (1-\eta_t)^{n-1-k} [P((n-1-k)q_t^I + kq_t^R + q_{t,i})q_{t,i} - C(q_{t,i})], \tag{28}$$

is maximized given the production of the other players and the population fraction. Here q_t^R is the symmetric output level of each of the other rational firms in period t, and q_t^I is the output level of each of the imitator firms in period t. The first order condition for an optimum is characterized by equality between marginal cost an *expected* marginal revenue.

Given the value of q_t^I and the fraction η_t, all rational firms coordinate on the same output level q_t^R. This gives the first order condition

$$\frac{\delta U_t^R(q_{i,t}|q_t^R, q_t^I, \eta_t)}{\delta q_{i,t}} = 0,$$

which equals to

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \eta_t^k (1-\eta_t)^{n-1-k} \times$$
$$[P((n-1-k)q_t^I + (k+1)q_t^R) + q_t^R P'((n-1-k)q_t^I + (k+1)q_t^R) - C'(q_t^R)] = 0. \tag{29}$$

Let the solution to equation (29) be given by $q_t^R = H^R(q_t^I, \eta_t)$, the full system of equations is thus given by

$$\begin{aligned} q_{t+1}^R &= H^R(q_{t+1}^I, \eta_{t+1}) \\ q_{t+1}^I &= \eta_t q_t^R + (1-\eta_t) q_t^I \\ \eta_{t+1} &= K(\Delta U_t). \end{aligned} \tag{30}$$

where $\Delta U_t = U_t^R - U_t^I$. It is easily checked that if the imitators play the Cournot-Nash equilibrium quantity q^*, or if all firms are rational, then the rational firms will play the Cournot-Nash equilibrium quantity, that is $H^R(q^*, \eta) = q^*$, for all η and $H^R(q^I, 1) = q^*$ for all q^I. Moreover, if a rational firm is certain it will only meet imitation firms (that is $\eta = 0$), it plays a best response to the currently average played quantity, that is $H^R(q_t^I, 0) = R((n-1)q_t^I)$, for all q_t^I. In the remainder we will denote the partial derivatives of $H^R(q, \eta)$ with respect to q and η by $H_q^R(q, \eta)$ and $H_\eta^R(q, \eta)$ respectively.

Proposition 4 *The Cournot-Nash equilibrium (q^*, q^*, η^*) is a locally stable fixed point for the model with endogenous fractions of rational and imitation firms, where all firms produce the Cournot-Nash quantity, if and only if*

$$|\eta^* H_q(q^*, \eta^*) + 1 - \eta^*| < 1 \tag{31}$$

Proof. Since a dynamical system can only depend on lagged variables, we substitute the second and third equation into the first. This gives us the following system that depends only on lagged variables.

$$\begin{aligned} q_{t+1}^R &= \psi^1 = H^R(\eta_t q_t^R + (1-\eta_t)q_t^I, K(\Delta U_t^R)) \\ q_{t+1}^I &= \psi^2 = \eta_t q_t^R + (1-\eta_t) q_t^I \\ \eta_{t+1} &= \psi^3 = K(\Delta U_t^R). \end{aligned} \tag{32}$$

In the equilibrium all firms produce the Cournot-Nash quantity q^*, therefore profits are equal, hence the equilibrium fraction is given by $\eta^* = K(-C)$. In order to determine the local stability of the equilibrium (q^*, q^*, η^*) where all firms produce the Cournot-Nash quantity, we need to determine the eigenvalues of the Jacobian matrix of system (19), evaluated at the equilibrium.

The partial derivatives of ψ^2 with respect to q_t^R, q_t^I and η_t, evaluated at the equilibrium are η^*, $1-\eta^*$ and 0 respectively.

Next, let us determine the partial derivatives of ψ^3 with respect to q_t^R, q_t^I and η_t, respectively. To that end, note that we can write the profit differential as

$$\Delta U_t^R = \Pi_t^R - \Pi_t^I - C = \sum_{k=0}^{n-1} A_k(\eta_t) D_k(q_t^R, q_t^I, \eta_t) - C,$$

with $A_k(\eta_t) = \binom{n-1}{k} \eta_t^k (1-\eta_t)^{n-1-k}$, which does not depend upon q^R and q^I, and

$$\begin{aligned} D_k(q_t^R, q_t^I, \eta_t) =& P((k+1)q^R + (n-1-k)q^I)q^R - C(q^R) \\ &- [P(kq^R + (n-k)q^I)q^I - C(q^I)], \end{aligned} \tag{33}$$

which depends upon η_t through $q_t^R = H(q_t^I, \eta_t)$. Note that $D_k(q_t^R, q^*, \eta_t) = 0$, moreover the partial derivatives of $D_k(q_t^R, q_t^I, \eta_t)$, evaluated at the equilibrium (q^*, q^*, η^*) are given by

$$\begin{aligned} \left.\frac{\delta D_k(q_t^R, q_t^I, \eta_t)}{\delta q_t^R}\right|_{(q^*, q^*, \eta^*)} &= [P'^*) q^* + P(Q^*) - C'^*)] H_q(q^*, \eta^*) = 0, \\ \left.\frac{\delta D_k(q_t^R, q_t^I, \eta_t)}{\delta q_t^I}\right|_{(q^*, q^*, \eta^*)} &= -[P'^*) q^* - P(Q^*) + C'^*)] = 0, \\ \left.\frac{\delta D_k(q_t^R, q_t^I, \eta_t)}{\delta \eta_t}\right|_{(q^*, q^*, \eta^*)} &= [P'^*) q^* P(Q^*) - C'^*)] H_\eta(q^*, \eta^*) = 0. \end{aligned}$$

The second equalities follows from the fact that $P'^{*})q^* + P(Q^*) - C'^{*}) = 0$ is the first order condition of any firm in a Cournot-Nash equilibrium. Furthermore $D_k(q^*, q^*, \eta) = 0$ for all η by the first order condition for a Cournot-Nash equilibrium. Using this it follows immediately that:

$$\begin{aligned}\left.\frac{\delta\psi^3}{\delta q_t^R}\right|_{(q^*,q^*,\eta^*)} &= K'(-C)\left.\frac{\delta\Delta U_t}{\delta q_t^R}\right|_{(q^*,q^*,\eta^*)} \\ &= K'(-C)\sum_{k=0}^{n-1} A_k(\eta^*)\left.\frac{\delta D_k(q_t^R, q_t^I, \eta_t)}{\delta q_t^R}\right|_{(q^*,q^*,\eta^*)} \\ &= 0\end{aligned} \tag{34}$$

and

$$\begin{aligned}\left.\frac{\delta\psi^3}{\delta q_t^I}\right|_{(q^*,q^*,\eta^*)} &= K'(-C)\left.\frac{\delta\Delta U_t}{\delta q_t^I}\right|_{(q^*,q^*,\eta^*)} \\ &= K'(-C)\sum_{k=0}^{n-1} A_k(\eta^*)\left.\frac{\delta D_k(q_t^R, q_t^I, \eta_t)}{\delta q_t^I}\right|_{(q^*,q^*,\eta^*)} \\ &= 0\end{aligned} \tag{35}$$

and

$$\begin{aligned}\left.\frac{\delta\psi^3}{\delta \eta_t}\right|_{(q^*,q^*,\eta^*)} &= K'(-C)\left.\frac{\delta\Delta U_t}{\delta \eta_t}\right|_{(q^*,q^*,\eta^*)} \\ &= K'(-C)\sum_{k=0}^{n-1}[A_k(\eta^*)\left.\frac{\delta D_k(q_t^R, q_t^I, \eta_t)}{\delta \eta_t}\right|_{(q^*,q^*,\eta^*)} + \frac{\delta A_k(\eta)}{\delta\eta_t} D_k(q^{R^*}, q_t^{I^*}, \eta^*)] \\ &= 0.\end{aligned} \tag{36}$$

This leaves us to examine the partial derivatives of ψ^1 with respect to q_t^R, q_t^I and η_t, evaluated at the equilibrium.

$$\begin{aligned}\left.\frac{\delta\psi^1}{\delta q_t^R}\right|_{(q^*,q^*,\eta^*)} &= \eta^* H_q^R(q^*, \eta^*) + \frac{\delta K(\Delta U_t)}{\delta q_t^R} H_\eta(q^*, \eta^*) \\ &= \eta^* H_q^R(q^*, \eta^*)\end{aligned} \tag{37}$$

and

$$\begin{aligned}\left.\frac{\delta\psi^1}{\delta q_t^I}\right|_{(q^*,q^*,\eta^*)} &= (1-\eta^*) H_q^R(q^*, \eta^*) + \frac{\delta K(\Delta U_t)}{\delta q_t^I} H_\eta(q^*, \eta^*) \\ &= (1-\eta^*) H_q^R(q^*, \eta^*)\end{aligned} \tag{38}$$

and

$$\left.\frac{\delta\psi^1}{\delta\eta_t}\right|_{(q^*,q^*,\eta^*)} = (q^*-q^*)H_q^R(q^*,\eta^*) + \frac{\delta K(\Delta U_t)}{\delta\eta_t}H_\eta(q^*,\eta^*) \\ = 0 \tag{39}$$

Therefore the Jacobian matrix, evaluated at the equilibrium is given by

$$J|_{q^*,q^*,\eta^*} = \begin{pmatrix} \eta^* H_q^R(q^*,\eta^*) & (1-\eta^*)H_q^R(q^*,\eta^*) & 0 \\ \eta^* & 1-\eta^* & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

Which has eigenvalues $\lambda_1 = \eta H_q(q*,\eta*) + 1 - \eta^*$, $\lambda_2 = 0$ and $\lambda_3 = 0$. Consequently the system is locally stable when $|\lambda_1| < 1$, this is exactly the condition stated in proposition 4. Note again the similarity with the condition in Section 3 where we fixed the fraction η.

Leading example. Since the stability condition is the similar to the condition derived in Section 3.2, the equilibrium (q^*,q^*,η^*) is stable for all n in this linear specification.

5 Rational vs. Cournot vs. Imitation

In this section we combine the ideas that we gathered in Section 4. We will investigate the dynamics when the three heuristics discussed before compete. As before every round n firms are drawn from a large pool of firms to play the one-shot Cournot game. From this large pool of firms a fraction η_t^R plays according to the rational strategy in period t, a fraction η_t^C plays according to the Cournot heuristic in period t and consequently the fraction of imitators in period t is determined by $1-\eta_t^R-\eta_t^C$. As in Section 4 the fitness of a heuristic is determined by the average payoff minus the information cost of using that heuristic. Again the average profits will be approximated by the expected profits but in contrast to Section 4 the distribution of states now follows a multinomial distribution instead of a binomial distribution. In general the average profit of a firm producing q_1 and competing with other firms that produce either q_2 or q_3 given the fractions η_1 and η_2 is stated below, in this average profit approximation the profit in each state is weighted by the chance of this state.

$$\Pi_{1,t} = F(q_{1,t},q_{2,t},q_{3,t},\eta_{1,t},\eta_{2,t}) = \\ \sum_{\Delta}\frac{(n-1)!}{k_1!k_2!(n-k_1-k_2-1)!}\eta_{1,t}^{k_1}\eta_{2,t}^{k_2}(1-\eta_{1,t}-\eta_{2,t})^{n-k_1-k_2-1}\times \\ P((k_1+1)q_{1,t}+k_2q_{2,t}+(n-1-k_1-k_2)q_{3,t})q_{1,t} - C(q_{1,t}), \tag{41}$$

The summation is over all possible combinations of k_1 and k_2, which stand for the number of other firms producing q_1 and q_2 respectively, that is: $\Delta = \{k_1,k_2 \in \mathbb{I}^2 : 0 \le k_1 \le n-1; 0 \le k_2 \le n-1; 0 \le k_1+k_2 \le n-1\}$. Expected profits for heuristic 2 in period t are given by $F(q_{2,t}, q_{1,t}, q_{3,t}, \eta_{2,t}, \eta_{1,t})$, expected profits for heuristic 3 in period t are given by $F(q_{3,t}, q_{2,t}, q_{1,t}, 1-\eta_{1,t}-\eta_{2,t}, \eta_{2,t})$.

The complete dynamical system consists of five equations, three for the quantity dynamics and two to describe how the fractions evolve. As in all previous sections, the Cournot firms play in the next period a best-response to the current aggregated output of the others, imitators play in the next period the average produced quantity by the others in the current period. Rational players produce every period the quantity that maximizes expected payoff given the fractions and production plans of all other firms (imitators, Cournot players but rational players too). The rational firms produce expectations over all possible mixtures of heuristics resulting from randomly drawing the $n-1$ other players from the large population of firms. In this setting the rational objective function, its own *expected* utility is of the following form:

$$\begin{aligned} U_t^R(q_{i,t}|x) = &\sum_{\Delta} f_{k_1,k_2}(n-1, \eta^R, \eta^C)\times \\ &P(k_1 q_t^R + k_2 q_t^C + (n-1-k_1-k_2)q_t^I + q_{i,t})q_{i,t} - C(q_{i,t}), \end{aligned} \tag{42}$$

with $f_{k_1,k_2}(n-1,\eta^R,\eta^C) = \frac{(n-1)!}{k_1!k_2!(n-k_1-k_2-1)!}{\eta_t^R}^{k_1}{\eta_t^C}^{k_2}(1-\eta_t^R-\eta_t^C)^{n-k_1-k_2-1}$ and $x = q_t^R, q_t^I, q_t^C, \eta_t^R, \eta_t^C$. The first order condition for an optimum of (42) is characterized by equality between marginal cost an *expected* marginal revenue.

Given the value of q_t^C q_t^I η_t^R η_t^C, all rational firms coordinate on the same output level q_t^R. Differentiating equation (42) with respect to $q_{i,t}$ gives the first order condition, which is equal for all rational firms. This first order condition is given by:

$$\frac{\delta U_t^R(q_{i,t}|x)}{\delta q_{i,t}} = 0$$

which equals to:

$$\begin{aligned} &\sum_{\Delta} f_{k_1,k_2}(n-1,\eta^R,\eta^C)\times \\ &[P((k_1+1)q_t^R + k_2 q_t^C + (n-1-k_1-k_2)q_t^I)+ \\ P'((k_1+1)q_t^R &+ k_2 q_t^C + (n-1-k_1-k_2)q_t^I)q_t^R - C'(q_t^R)] = 0 \end{aligned} \tag{43}$$

Let the solution to this be given by $q_t^R = H^R(q_t^C, q_t^I, \eta_t^R, \eta_t^C)$. The system of quantity dynamics is thus given by

$$\begin{aligned} q_{t+1}^R &= H^R(q_{t+1}^C, q_{t+1}^I, \eta_{t+1}^R, \eta_{t+1}^C) \\ q_{t+1}^C &= R((n-1)(\eta_t^R q_t^R + \eta_t^C q_t^C + (1-\eta_t^R-\eta_t^C)q_t^I) \\ q_{t+1}^I &= \eta_t^R q_t^R + \eta_t^C q_t^C + (1-\eta_t^R-\eta_t^C)q_t^I \end{aligned} \tag{44}$$

Note that rational player plays such that *expected* marginal revenue equals marginal cost at $t+1$ and a Cournot firm plays such that its marginal revenue (of period t) equals marginal cost (at period t). Therefore the Cournot heuristic is a lagged version of the rational heuristic if and only if

$$\begin{aligned} P'^R q^R + \eta^C q^C + (1-\eta^R-\eta^C)q^I) + q^C) = \\ \sum_{\Delta} f_{k_1,k_2}(n-1,\eta^R,\eta^C)P'((k_1+1)q^R + k_2 q^C + (n-1-k_1-k_2)q^I). \end{aligned} \tag{45}$$

Thus the Cournot heuristic is only a lagged version of the rational heuristic if the inverse demand is linear. In this specific case the analysis become easier because this gives the possibility to lower the dimension of the dynamical system.

It is easily checked that if the imitation and Cournot firms play the Cournot-Nash equilibrium quantity q^*, or if all firms are rational, the rational firms will play the Cournot-Nash equilibrium quantity, that is $H^R(q^*, q^*, \eta_t^R, \eta_t^C) = q^*$, for all η^R and η^C and $H^R(q_{t+1}^C, q_{t+1}^I, 1, 0) = q^*$ for all q^C, q^I. In the remainder we will denote by $H^R_{q^R}$, $H^R_{q^C}$, $H^R_{q^I}$, $H^R_{\eta^R}$ and $H^R_{\eta^C}$ the partial derivatives of $H^R(q^C, q^I, \eta^R, \eta^C)$ with respect to q^R, q^C, q^I, η^R and η^C respectively, evaluated at the equilibrium $(q^*, q^*, q^*, \eta^{R^*} \eta^{C^*})$, which we will denote by x^* in the remainder of this chapter for notational convenience.

Now that we have the quantity dynamics we can turn to the population dynamics. These are related to the population dynamics from Section 4 but differ significantly since we are in a three heuristic environment now. The population dynamics, as in Section 4, depend on *relative* fitness. Let the fraction dynamics be given by

$$\begin{aligned} \eta_{R,t+1} &= K^R(\Delta U_t^R, \Delta U_t^C) \\ \eta_{C,t+1} &= K^C(\Delta U_t^R, \Delta U_t^C). \end{aligned} \tag{46}$$

Where $\eta_{t+1}^R$is the fraction of rational firms in period $t+1$ whereas η_{t+1}^C is the fraction of Cournot firms in that period. With $\Delta U_t^R = \Pi_t^R - C^R - (\Pi_t^C - C^C)$ we denote the difference in average fitness of the rational and the Cournot heuristic and with $\Delta U_t^C = \Pi_t^C - C^C - (\Pi_t^I - C^I)$ we denote the difference in average fitness of the Cournot and the imitation heuristic. Note that K^R and K^C are $\mathbb{R}^2 \to [0,1]$ are continuously differentiable functions where the difference in fitness of the rational and Cournot heuristics and the difference in fitness of the Cournot and imitation heuristic are used as input. The difference in fitness of the rational and imitation heuristic is not used as an input variable since this information is captured implicitly in the other two differences. Note that K^R is a monotonically increasing function in the first and second element whereas K^C is decreasing in the first element but increasing in the second element. Furthermore, $K^R(0,0) = K^C(0,0) = \frac{1}{3}$. In the remainder of this chapter we denote K_1^R and K_2^R the partial derivatives of K^R with respect to the first and the second element respectively and with K_1^C and K_2^C the partial derivatives of K^C with respect to the first and the second element respectively.

Now that we have the quantity and population dynamics, we know the full system of equations. The full system is given by:

$$\begin{aligned}
q^R_{t+1} &= \phi^1 = H^R(\phi^2,\phi^3,\phi^4,\phi^5)\\
q^C_{t+1} &= \phi^2 = R((n-1)(\eta^R_t H^R(q^C_t,q^I_t,\eta^R_t,\eta^C_t) + \eta^C_t q^C_t + (1-\eta^R_t-\eta^C_t)q^I_t)\\
q^I_{t+1} &= \phi^3 = \eta^R_t q^R_t + \eta^C_t q^C_t + (1-\eta^R_t-\eta^C_t)q^I_t\\
\eta^R_{t+1} &= \phi^4 = K^R(\Delta U^R_t,\Delta U^C_t)\\
\eta^C_{t+1} &= \phi^5 = K^C(\Delta U^R_t,\Delta U^C_t).
\end{aligned} \tag{47}$$

Since a dynamical system can only depend on lagged variables we substituted $\phi^2,\phi^3,\phi^4,\phi^5$ into $H^R(\cdot)$. In order to determine the local stability of the unique equilibrium x^*, we need to determine the eigenvalues of the Jacobian matrix evaluated at that equilibrium x^*.

It can easily be shown that the partial derivatives of ϕ^3 with respect to q^R, q^C, q^I, η^R and η^C, evaluated at the equilibrium are η^{R^*}, η^{C^*}, $1-\eta^{R^*}-\eta^{C^*}$, 0 and 0 respectively.

To determine the partial derivatives of ϕ^4 and ϕ^5 we need to determine the partial derivatives of ΔU^R_t and ΔU^C_t. In accordance to Section 4.2 we can write the first profit differential as

$$\Delta U^R = \sum_{\Delta} A_k(\eta^R,\eta^C) D_k(q^R_t,q^C_t,q^I_t,\eta^R,\eta^C) - C^R + C^C,$$

with $A_{k_1,k_2}(\eta^R_t,\eta^C_t) = \frac{(n-1)!}{k_1!k_2!(n-k_1-k_2-1)!}{\eta^R_t}^{k_1}{\eta^C_t}^{k_2}(1-\eta^R_t-\eta^C_t)^{n-k_1-k_2-1}$, which does not depend upon the produced quantities, and

$$\begin{aligned}
D_{k_1,k_2}(q^R_t,q^C_t,q^I_t,\eta^R_t,\eta^C_t) &= P((k_1+1)q^R_t + k_2 q^C_t + (n-k_1-k_2-1)q^I_t)q^R_t - C(q^R_t)\\
&\quad - [P(k_1 q^R_t + (k_2+1)q^C_t + (n-k_1-k_2-1)q^I_t)q^C_t - C(q^C_t)].
\end{aligned} \tag{48}$$

Which depends upon η^R and η^C through $q^R_t = H^R(q^C_t,q^I_t,\eta^R_t,\eta^C_t)$. Note that $D_{k_1,k_2}(q^*,q^*,q^*,\eta^R_t,\eta^C_t) = 0,\ \forall\,\eta^R,\eta^C$. Next to that the partial derivatives of $D_{k_1,k_2}(x)$ evaluated at the equilibrium are given by

$$\begin{aligned}
\left.\frac{\delta D_k(q^R_t,q^C_t,q^I_t,\eta^R,\eta^C)}{\delta q^R_t}\right|_{x^*} &= [P'^*)q^* + P(Q^*) - C'^*)]H^R_{q^R}(x^*) = 0\\
\left.\frac{\delta D_k(q^R_t,q^C_t,q^I_t,\eta^R,\eta^C)}{\delta q^C_t}\right|_{x^*} &= [P'^*)q^* + P(Q^*) - C'^*)](H^R_{q^C}(x^*) - 1) = 0\\
\left.\frac{\delta D_k(q^R_t,q^C_t,q^I_t,\eta^R,\eta^C)}{\delta q^I_t}\right|_{x^*} &= [P'^*)q^* + P(Q^*) - C'^*)]H^R_{q^I}(x^*) = 0\\
\left.\frac{\delta D_k(q^R_t,q^C_t,q^I_t,\eta^R,\eta^C)}{\delta \eta^R}\right|_{x^*} &= [P'^*)q^* + P(Q^*) - C'^*)]H_{\eta^R}(x^*) = 0\\
\left.\frac{\delta D_k(q^R_t,q^C_t,q^I_t,\eta^R,\eta^C)}{\delta \eta^C}\right|_{x^*} &= [P'^*)q^* + P(Q^*) - C'^*)]H_{\eta^C}(x^*) = 0.
\end{aligned} \tag{49}$$

Where the second equalities follow from the fact that $P'^{*})q^{*}+P(Q^{*})-C'^{*})=0$ is the first order condition of any firm in a Cournot-Nash equilibrium. Using this it follows immediately that the partial derivatives of ϕ^4 are given by:

$$\left.\frac{\delta\phi^4}{\delta q_t^R}\right|_{x^*} = K_1^R(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^R}{\delta q_t^R}\right|_{x^*} + K_2^R(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^C}{\delta q_t^R}\right|_{x^*} = 0 \tag{50}$$

and

$$\left.\frac{\delta\phi^4}{\delta q_t^C}\right|_{x^*} = K_1^R(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^R}{\delta q_t^C}\right|_{x^*} + K_2^R(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^C}{\delta q_t^C}\right|_{x^*} = 0 \tag{51}$$

and

$$\left.\frac{\delta\phi^4}{\delta q_t^I}\right|_{x^*} = K_1^R(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^R}{\delta q_t^I}\right|_{x^*} + K_2^R(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^C}{\delta q_t^I}\right|_{x^*} = 0 \tag{52}$$

and

$$\left.\frac{\delta\phi^4}{\delta \eta_t^R}\right|_{x^*} = K_1^R(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^R}{\delta \eta_t^R}\right|_{x^*} + K_2^R(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^C}{\delta \eta_t^R}\right|_{x^*} = 0 \tag{53}$$

and

$$\left.\frac{\delta\phi^4}{\delta \eta_t^C}\right|_{x^*} = K_1^R(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^R}{\delta \eta_t^C}\right|_{x^*} + K_2^R(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^C}{\delta \eta_t^C}\right|_{x^*} = 0. \tag{54}$$

Furthermore, the partial derivatives of ϕ^5 are given by

$$\left.\frac{\delta\phi^5}{\delta q_t^R}\right|_{x^*} = K_1^C(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^R}{\delta q_t^R}\right|_{x^*} + K_2^C(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^C}{\delta q_t^R}\right|_{x^*} = 0 \tag{55}$$

and

$$\left.\frac{\delta\phi^5}{\delta q_t^C}\right|_{x^*} = K_1^C(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^R}{\delta q_t^C}\right|_{x^*} + K_2^C(C^C-C^R, C^I-C^C)\left.\frac{\delta\Delta U^C}{\delta q_t^C}\right|_{x^*} = 0 \tag{56}$$

and

$$\left.\frac{\delta\phi^5}{\delta q_t^I}\right|_{x^*} = K_1^C(C^C - C^R, C^I - C^C)\left.\frac{\delta\Delta U^R}{\delta q_t^I}\right|_{x^*} + K_2^C(C^C - C^R, C^I - C^C)\left.\frac{\delta\Delta U^C}{\delta q_t^I}\right|_{x^*}$$
$$= 0 \tag{57}$$

and

$$\left.\frac{\delta\phi^5}{\delta \eta_t^R}\right|_{x^*} = K_1^C(C^C - C^R, C^I - C^C)\left.\frac{\delta\Delta U^R}{\delta \eta_t^R}\right|_{x^*} + K_2^C(C^C - C^R, C^I - C^C)\left.\frac{\delta\Delta U^C}{\delta \eta_t^R}\right|_{x^*}$$
$$= 0 \tag{58}$$

and

$$\left.\frac{\delta\phi^5}{\delta \eta_t^C}\right|_{x^*} = K_1^C(C^C - C^R, C^I - C^C)\left.\frac{\delta\Delta U^R}{\delta \eta_t^C}\right|_{x^*} + K_2^C(C^C - C^R, C^I - C^C)\left.\frac{\delta\Delta U^C}{\delta \eta_t^C}\right|_{x^*}$$
$$= 0. \tag{59}$$

The Jacobian of the system, evaluated at the equilibrium x^* is thus given by

$$J|_{x^*} = \begin{pmatrix} H^R_{q^R} & H^R_{q^C} & H^R_{q^I} & H^R_{\eta^R} & H^R_{\eta^C} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} \\ \eta^{R^*} & \eta^{C^*} & 1-\eta^{R^*}-\eta^{C^*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{60}$$

with

$$J_{21} = (n-1)\eta^{R^*}H^R_{q^R}R'^*)$$
$$J_{22} = (n-1)\left(\eta^{R^*}H^R_{q^C} + \eta^{C^*}\right)R'^*)$$
$$J_{23} = (n-1)\left(\eta^{R^*}H^R_{q^I} + 1 - \eta^{R^*} - \eta^{C^*}\right)R'^*)$$
$$J_{24} = (n-1)\left(H^R\big|_{x^*} + \eta^{R^*}H^R_{\eta^R} - q^*\right)R'\left((n-1)q^*\right)$$
$$J_{25} = (n-1)\left(H^R\big|_{x^*} + \eta^{R^*}H^R_{\eta^C} - q^*\right)R'\left((n-1)q^*\right).$$

This Jacobian has very complicated eigenvalues which cannot be expressed in a useful function, for this we have to look at the leading example.

Leading example. We know that the Cournot heuristic is a lagged version of the rational heuristic in this leading example since the inverse demand function is linear, therefore the dimension of the dynamical system can be reduced by one. Note that only the Cournot *production* is a lagged version of the rational *production*. The Cournot profits and resulting fractions are in general not lagged rational profits and fractions. The production plans of the system, are given by

$$\begin{aligned}
q_{t+1}^R &= H^R(x_t) = \frac{a-c}{b(2+(n-1)\eta_{t+1}^R)} - \frac{n-1}{2+(n-1)\eta_{t+1}^R}(\eta_{t+1}^C q_{t+1}^C + (1-\eta_{t+1}^R - \eta_{t+1}^C)q_{t+1}^I) \\
q_{t+1}^C &= \frac{a-c}{2b} - \frac{1}{2}(n-1)(\eta_t^R q_t^R + \eta_t^C q_t^C + (1-\eta^R - \eta^C)q_t^I) \\
q_{t+1}^I &= \eta_t^R q_t^R + \eta_t^C q_t^C + (1-\eta^R-\eta^C)q_t^I,
\end{aligned} \tag{61}$$

where $x_t = (q_t^C, q_t^I, \eta_t^R, \eta_t^C)$. Furthermore, the average profit (Eq. (5)) boils in the leading example down to

$$\begin{aligned}
\Pi_t^R =& F(H^R(x_t), q_t^C, q_t^I, \eta_t^R, \eta_t^C) \\
=& (a-c)H^R(x_t) - b(H^R(x_t) + (n-1)q_t^I)H^R(x_t) \\
& - b\left(H^R(x_t) - q_t^I\right)(n-1)\eta_t^R H^R(x_t) - b(q_t^C - q_t^I)(n-1)\eta_t^C H^R(x_t),
\end{aligned} \tag{62}$$

using that

$$\begin{aligned}
\sum_{\Delta} \frac{(n-1)!}{k_1!k_2!(n-k_1-k_2-1)!} {\eta_t^R}^{k_1} {\eta_t^C}^{k_2} (1-\eta_t^R-\eta_t^C)^{n-k_1-k_2-1} &= 1, \\
\sum_{\Delta} \frac{(n-1)!}{k_1!k_2!(n-k_1-k_2-1)!} {\eta_t^R}^{k_1} {\eta_t^C}^{k_2} (1-\eta_t^R-\eta_t^C)^{n-k_1-k_2-1} k_1 &= (n-1)\eta_t^R, \\
\sum_{\Delta} \frac{(n-1)!}{k_1!k_2!(n-k_1-k_2-1)!} {\eta_t^R}^{k_1} {\eta_t^C}^{k_2} (1-\eta_t^R-\eta_t^C)^{n-k_1-k_2-1} k_2 &= (n-1)\eta_t^C.
\end{aligned} \tag{63}$$

Remember that $\Pi_t^C = F(q_t^C, H^R(x_t), q_t^I, \eta_t^C, \eta_t^R)$ and $\Pi_t^I = F(q_t^I, q_t^C, H^R(x_t), 1-\eta_t^R-\eta_t^C, \eta_t^C)$. For the population dynamics we use the Logit dynamics, as for example discussed in Brock and Hommes (1997). The complete dynamical system in this leading example is thus given by [4]

$$
\begin{aligned}
q^C_{t+1} &= \phi^1(x_t) = \frac{1}{2+\eta^R(n-1)}\left(\frac{a-c}{2b} - \frac{1}{2}(n-1)(\eta^C_t q^C_t + (1-\eta^R-\eta^C)q^I_t)\right)\\
q^I_{t+1} &= \phi^2(x_t) = \eta^R_t\phi^1(x) + \eta^C_t q^C_t + (1-\eta^R-\eta^C)q^I_t\\
\eta^R_{t+1} &= \phi^3(x_t) = \frac{e^{\beta\Delta U^R_t}}{e^{\beta\Delta U^R_t} + 1 + e^{-\beta\Delta U^C_t}}\\
\eta^C_{t+1} &= \phi^4(x_t) = \frac{e^{\beta\Delta U^C_t}}{e^{\beta(\Delta U^R_t+\Delta U^C_t)} + e^{\beta\Delta U^C_t} + 1}.
\end{aligned}
\tag{64}
$$

This system has one unique equilibrium where all firms produces the Cournot-Nash quantity q^*. Since production is equal at the equilibrium, profits are equal at the equilibrium. The equilibrium fractions are therefore a function of the information costs and the evolutionary pressure, given by

$$
\eta^{R^*} = \frac{e^{\beta(C^C-C^R)}}{e^{\beta(C^C-C^R)} + 1 + e^{-\beta(C^I-C^C)}} \text{ and } \eta^{C^*} = \frac{e^{\beta(C^I-C^C)}}{e^{\beta(C^I-C^R)} + e^{-\beta(C^I-C^C)} + 1}.
$$

The Jacobian of the system in the leading example evaluated at the equilibrium is therefore given by

$$
J\big|_{x^*} = \begin{pmatrix}
-\frac{(n-1)\eta^{C^*}}{2+\eta^{R^*}(n-1)} & -\frac{(n-1)(1-\eta^{R^*}-\eta^{C^*})}{2+\eta^{R^*}(n-1)} & J_{13} & J_{14}\\
\eta^{C^*}\left(1-\frac{(n-1)\eta^{R^*}}{2+\eta^{R^*}(n-1)}\right) & (1-\eta^{R^*}-\eta^{C^*})\left(1-\frac{(n-1)\eta^{R^*}}{2+\eta^{R^*}(n-1)}\right) & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{pmatrix},
\tag{65}
$$

For the calculation of the eigenvalues J_{13} and J_{14} are irrelevant because the third and fourth row contain only zeros. For general η^{R^*} and η^{C^*} the eigenvalues become very lengthy expressions which cannot be simplified. Nevertheless, because the rational heuristic uses much more information than the Cournot and the imitation heuristic, we set the cost of this heuristic equal to $C^R > 0$, while we set the cost of the other heuristics equal to 0, without loss of generality. After this parameterization we can calculate the eigenvalues analytically. The eigenvalues are a function of n and the the product of β and C^R. The eigenvalues are given by

$$
\lambda_1 = \lambda_2 = \lambda_3 = 0 \text{ and } \lambda_4(n, C^R\beta) = \frac{3e^{C^R\beta} - ne^{C^R\beta}}{n + 4e^{C^R\beta} + 1}
$$

[4] The population dynamics can alternatively be expressed as

$$
\begin{aligned}
\eta^R_{t+1} &= \frac{e^{\beta U_{R,t}}}{e^{\beta U_{R,t}} + e^{\beta U_{C,t}} + e^{\beta U_{I,t}}}\\
\eta^C_{t+1} &= \frac{e^{\beta U_{C,t}}}{e^{\beta U_{R,t}} + e^{\beta U_{C,t}} + e^{\beta U_{I,t}}},
\end{aligned}
$$

which is a more common but equivalent expression.

The rational production plan is left out of the dynamical system because the Cournot production plan is a lagged version of the rational production plan.

For the system to be stable we need $|\lambda_4(n, C^R\beta)| < 1$. Note that λ_4 is always less than 1. Rearranging gives that the threshold number of firms is given by

$$n < \psi(C^{R^*}\beta) = \frac{7e^{C^R\beta} + 1}{e^{C^R\beta} - 1} \tag{66}$$

Note that economically the number of firms can only be an integer but mathematically the number of firms can be treated as a continuous variable.

From equation (66) we see that when information cost C^R is close to zero, the system is stable for all n. When $C^R = 1$ and $\beta = 3$, as simulated below, the equilibrium $\left(\frac{a-c}{b(n+1)}, \frac{a-c}{b(n+1)}, \frac{e^{-3}}{e^{-3}+2}, \frac{1}{e^{-3}+2}\right)$ is stable when $n < 7.42$. When $n = 7.42$, the system undergoes its first bifurcation. The largest eigenvalue is equal to -1 at the bifurcation, indicating that the first bifurcation is a period-doubling bifurcation. This is confirmed by the simulations below.

The leading example is simulated in the Fig. 4. Panel (a) depicts the bifurcation diagram for increasing number of firms n. The first period-doubling bifurcation appears, as calculated analytically for $n = 7.42$. For $n = 11.85$, the system undergoes a Hopf-bifurcation which creates highly non-linear dynamics. For $13 \leq n \leq 14.4$, the system is in a 10-cycle whereas for $n > 14.4$ the system becomes chaotic again. Panel (b) displays oscillating time series of produced quantity by the Cournot and imitation firms and the equilibrium quantity fraction q^*. Since the rational quantity in period $t+1$ equals the Cournot quantity in period t this time series is not included. Panel (c) displays the resulting profits. Note that $\Pi_t^I > \Pi_t^R\ \forall t$ and $\Pi_t^I \geq \Pi_t^C\ \forall t$. Panel (d) displays the resulting oscillating time series of the fraction fractions. Due to the information cost the sophisticated rational firms do not perform better than the Cournot and imitation firms resulting in low fractions of rational firms. Moreover, since the imitation profit is at least as high as the Cournot profit, the resulting imitation fraction is at least as high as the Cournot fraction. In Panel (e) the largest Lyapunov exponent for increasing number of firms is shown whereas in Panel (f) the largest Lyapunov exponent for increasing β is shown. Game and behavioural parameters are set equal to: $n = 19$, $a = 17$, $b = 1$, $c = 1$, $C^R = 1$, $C^C = 0$, $C^I = 0$, $\beta = 3$. Initial conditions are set equal to: $q_0^R = 0.3$, $q_0^C = 0.1$, $q_0^I = 0.25$, $\eta_0^R = 0.5$, $\eta_0^C = 0.2$. Last, figure shows some attractors of the evolutionary model for increasing evolutionary pressure, with (quasi-)periodic motion just after the second bifurcation and breaking of the invariant circles into a strange attractor as the number of firms further increases. Similar 'breaking of the invariant circles' route to chaos appears for the rational and Cournot series.

tncation

Figure 5: *Phaseplots* $(q^I, 1 - \eta^R - \eta^C)$*, for increasing evolutionary pressure.*

6 Concluding Remarks

In this paper we have investigated and generalized the work of Hommes, Ochea and Tuinstra (2011) on the stability of the Cournot-Nash equilibrium in a n-

firm, quantity-setting game. Hommes, Ochea and Tuinstra (2011) focus on competition between the Cournot adjustment process and the Nash quantity and competition between the Cournot adjustment process and rational production plans. In this thesis focus lays on the the imitation rule in competition with the Cournot and/or rational heuristics. We derive local stability conditions for our evolutionary model and find that introducing heterogeneity in heuristics tends to stabilize the dynamics, but chaotic behaviour is still very well possible.

In particular, with a linear inverse demand and constant marginal costs we find the following. Theocharis (1960) found that the dynamics are stable for $n < 3$. Hommes, Ochea and Tuinstra (2011) show that in the evolutionary competition between Cournot and the Nash heuristic the dynamics are stable for $n < 5$. Next to that Hommes, Ochea and Tuinstra (2011) show that in the absence of information costs for the rational heuristic, the Cournot-Nash equilibrium is stable for any $n \geq 2$ when there is evolutionary competition between the rational and the Cournot heuristic. If information cost are for the sophisticated heuristic strictly positive, the Cournot-Nash equilibrium becomes unstable if either the number of firms n, or the evolutionary pressure, as measured by β, increases.

When imitators compete with heterogeneous agents we found a similar result whereas in the case when imitators compete with either Cournot firms or rational firms we found differences namely, (i) in the case when Cournot firms compete with imitators we found that, in the absence of information costs, the threshold on the number of firms that changes the system from stable to unstable is 7, (ii) in the case when rational firms compete with imitators we found that the system is always stable, regardless of the information costs of the rational heuristic, (iii), in the case when rational firms, Cournot firms and imitators compete we found that the stability threshold on the number of firms depends on the evolutionary pressure and the stability of the cheapest heuristic(s). First, if the evolutionary pressure increases, the threshold on the number of firms for the system to be stable decreases. Second, if the cheapest behavioural rule is stable, the dynamics converge to a situation where most firms use this behavioural rule and all firms produce the Cournot-Nash equilibrium quantity. Next, if the cheapest heuristic is unstable, complicated endogenous fluctuations may occur. In particular, when the evolutionary pressure is high. Note that the nonlinearity causing this erratic behaviour comes from the endogenously updating of the fractions, because in our leading example the specifications were linear.

We decided to focus on the imitation, Cournot and rational heuristics, since there is no theoretical study on the evolutionary competition between these heuristics while there is experimental evidence that these heuristics are used in practice (Huck (2002)) .

The main contribution of this paper to the literature is to a broader knowledge of dynamical systems. This is to our best knowledge the first paper that investigates the stability of a model where three behavioural rules compete. We have used the Cournot oligopoly game, but in general any game with a continuous action space could be used to get the main points across, which are: (i) the importance for evolutionary game theory to consider games with a continuous action space, (ii) the focus on evolutionary competition between behavioural rules instead of competition between actions and (iii) the observation that endogenous fluctuations are a generic feature of evolutionary models. Moreover, this paper contributes to a better understanding of bounded rationality, with this understanding better expectations can be made.

For future research it would be interesting to focus on other interaction structures. Imagine that all firms are all located elsewhere, and whereas in the studied structure we assume that all firms observe all profits, in this setting we could for example assume that it only observes the profit of the m closest neighbors.

Other interesting extensions of this model are allowance of multiple equilibria or the introduction of other behavioural rules that coordinate on a 'non-Nash'-equilibrium, such as the cartel solution or the Walrasian equilibrium.

Experimental research confirms the findings in this thesis, making the results robust. Huck (2002) found that quantities do not explode when the quantity dynamics is described by a combination of the Cournot and the imitation heuristic. Since there are no information cost in Huck (2002), this result is in line with the findings in Section 4. Moreover, Bosch-Domènech, A., and N.J. Vriend (2003) find that the players do not rely more on imitation in more demanding environments and explain the different output decisions as predominantly relate to a general disorientation of the players, and more specifically to a significant decrease of best responses. This confirms that the choice of an evolutionary dynamic based on profits is justified and that the investigated heuristics in Section 5 are observed in experiments.

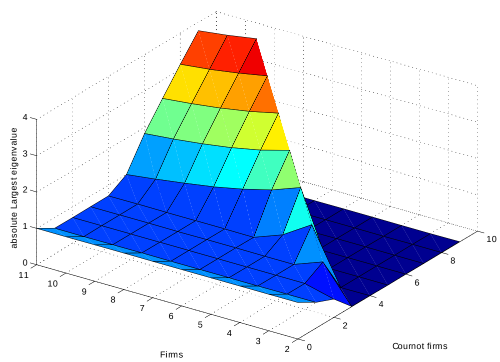
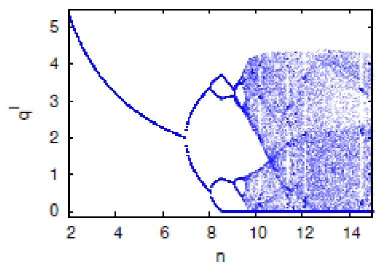
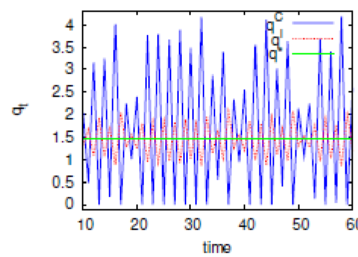


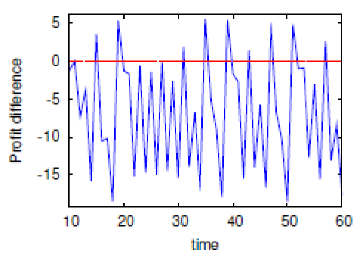
Figure 1: *Largest eigenvalue for the model rational vs. imitation firms. The largest eigenvalue decreases when the number of firms increases and when the fraction of rational players increases. Since an economy consisting of only imitation firms is neutrally stable, this model is stable for all combinations of* η *and* n.



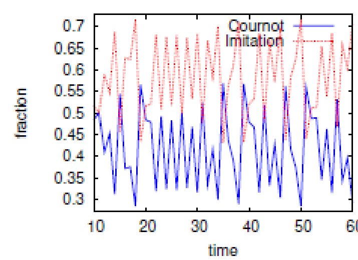
(a) Bifurcation diagram (q_t, n)



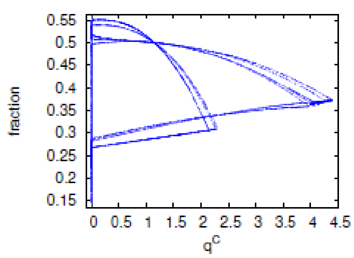
(b) Time path of Cournot and imitation quantities



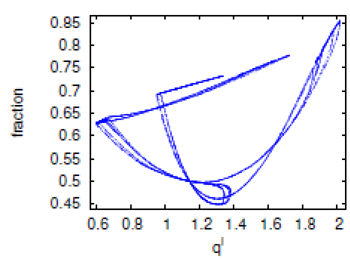
(c) Cournot profit differential



(d) Time path Cournot fraction



(e) Cournot phase plot



(f) Imitation phase plot

Figure 2: *Linear n-player Cournot game with endogenous fraction dynamics.*

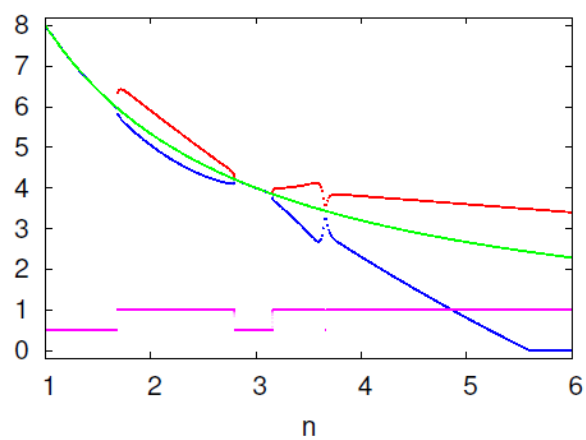
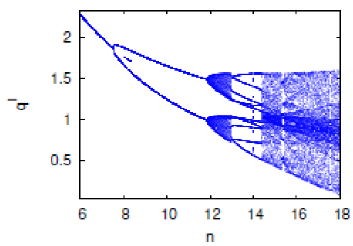
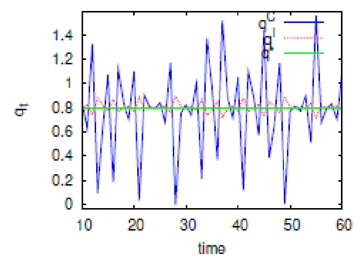


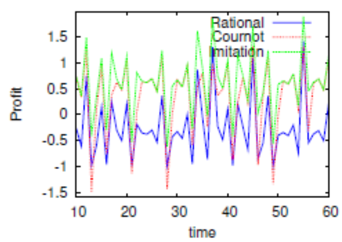
Figure 3: *Bifurcation diagram* (q_t, n) *with* $\beta = 3$



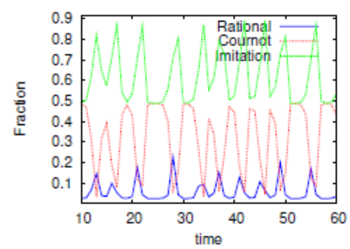
(a) Bifurcation diagram (q_t, n)



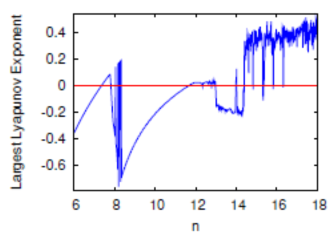
(b) Time path of Cournot and imitiation quantities



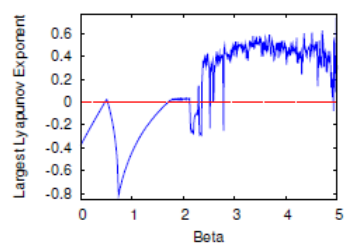
(c) Cournot profit differential



(d) Time path Cournot fraction

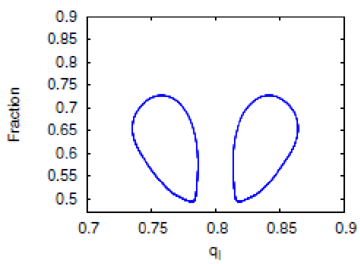


(e) Largest Lyapunov exponent

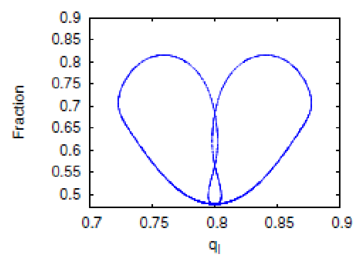


(f) Largest Lyapunov exponent

Figure 4: *Linear n-player Cournot competition between rational, Cournot and imitation firms with endogenous fraction dynamics.*

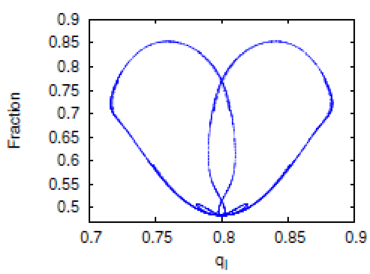


(a) phaseplot $\beta = 2$

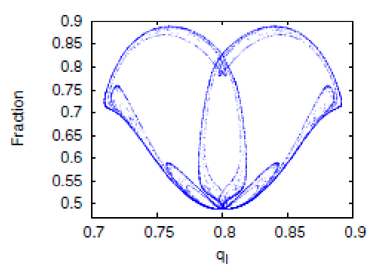


(b) phaseplot $\beta = 2.4$

ion



(c) phaseplot $\beta = 2.65$



(d) phaseplot $\beta = 2.95$